NASA TECHNICAL TRANSLATION NASA TT F-15,909

THE ATMOSPHERE AND THE CONTROL OF FLIGHT VEHICLE MOTION

Ye. P Shkol'nyy and L. A. Mayboroda

(NASA-TT-F-15909) THE ATMOSPHERE AND THE CONTROL OF FLIGHT VEHICLE MOTION (Kanner (Leo) Associates) 326 p

N74-33323

CSCL 22A

Unclas

G3/31 48742

Translation of Atmosfera i upravleniye dvizheniyem letatel'nykh apparatov, "Gidrometeoizdat" Press, Leningrad, 1973, 307 pp.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 SEPTEMBER 1974

325

STANDARD TITLE PAGE

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TT F-15,909		
4. Title and Subtitle THE ATMOSPHERE AND THE CONTROL OF FLIGHT VEHICLE MOTION		5. Report Date September 1974 6. Performing Organization Code
7. Author(s) Ye. P. Shkol'nyy and L. A. Mayboroda		8. Performing Organization Report No. 10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		11. Contract or Grant No. NASw-2481 13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of Atmosfera i upravelniye dvizheniyem letatel'nykh apparatov, "Gidrometeoizdat" Press, 1973, 307 pp.

16. Abstract

An analysis is given of the stochastic characteristics of the physical parameters of the atmosphere in the lower 100 km layer based on a statistical treatment of rocket sounding data. Models of the random components of the physical parameters of the atmosphere are developed; they are used in investigating the control of flight vehicle motion in the earth's atmosphere. Stochastic models of flight vehicle motion are examined and methods of the statistical analysis of the scatter of trajectories are proposed, along with methods of evaluating the effect of atmospheric perturbations on flight vehicle motion and methods of the statistical optimization of flight vehicle control systems.

17. Key Words (Selected by Author(s))	18. Distribution Statement
	Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	324	

PRECEDING PAGE BLANK NOT FILMED

NASA-HQ

AUTHORS' ABSTRACT /2

The book analyzes stochastic characteristics of the physical parameters of the atmosphere in the lower 100 km layer based on statistical treatment of rocket sounding data. Models of the random components of the physical parameters of the atmosphere are developed; they are used in investigating the control of flight vehicle motion in the earth's atmosphere. Stochastic models of flight vehicle motion are examined and methods of the statistical analysis of the scatter of trajectories are proposed, along with methods of evaluating the effect of atmospheric perturbations on flight vehicle motion and methods of the statistical optimization of flight vehicle control systems.

The book is addressed to engineers, graduate students, scientific coworkers, instructors, and students at hydrometeorological and technical high educational institutions concerned with problems of the physics and structure of the dense atmospheric layers and problems of flight vehicle motion in the earth's atmosphere.

EDITOR'S FOREWORD 3

The continuous expansion in the range of altitudes and velocities of control flight vehicle has meant that the interaction of flight vehicle with the ambient medium during flights in the atmosphere is becoming ever more complex. Therefore in solving problems of flight dynamics of controlled flight vehicle, increasingly fuller knowledge of the physical properties of the atmosphere and the correlations of their variation is needed. Accordingly, it became necessary to generalize, based on modern requirements, the wealth of accumulated experimental material.

Investigation of the motion of flight vehicles in the atmosphere, and especially, the solution of problems in optimizing the control of flight vehicle motion, depends not only on the completeness of information concerning atmospheric properties, but increasingly on the form in which this information is presented.

This monograph presents atmospheric characteristics in a form that is most convenient for the use of modern mathematical apparatus in solving problems of analyzing the dynamics of flight vehicle control and in synthesizing the optimal control of their motion on digital computers. In this respect, the monograph must be of great interest to a wide range of scientific workers, engineers, and graduate students in the field of the theory and practice of designing flight vehicle control systems. Specialist meteorologists may find extremely useful both the latest methods of solving the most complicated problems of analyzing and synthesizing nonlinear stochastic dynamic systems presented systematically in the monograph, as well as a thorough presentation of material on methods of representing random components of the thermodynamic parameters of the atmosphere.

The collaboration of two specialists in related fields of science -- meteorology and flight vehicle control theory -- unquestionably promoted a higher scientific level and greater applied orientation of the monograph. All this enables one to hope that the monograph will prove useful to a wide range of readers.

V. M. Ponomarev

AUTHORS' FOREWORD

/4

Among the numerous problems arising in recent years related to the mastery of near-earth space and the rapid development of flight vehicles, the problem of controlling flight vehicle motion in the dense atmospheric layers is vital.

The analysis and synthesis of systems for controlling flight vehicle motion in the dense layers of the atmosphere with allowance for the random scatter of its physical parameters relative to their nominal values determined by models of the standard atmosphere has led to the necessity of solving two interrelated problems.

On the one hand, we have the problem of determining the statistical characteristics of the random components of the physical parameters of the dense atmospheric layers and developing models and forms of representing the resulting information.

On the other hand, we have the problem of devising methods and algorithms for taking account of the effect of random components of the physical parameters of the atmosphere when solving the problem of analyzing the scatter of flight vehicle trajectory and the problem of synthesizing control systems of flight vehicle motion that provide the required precision in maintaining specified flight trajectories.

Success in solving the problems of analyzing and synthesizing control systems of flight vehicle motion is to a large extent determined by the form and the precision of specifying the random components of the physical parameters of the atmosphere in mathematical models of flight vehicle motion. Therefore in the monograph presented to the reader the authors attempted to collect and generalize extensive material of radiosonde, rocket, and radiometeor physical parameters of atmospheric measurements, to determine the statistical characteristics of random components of the physical parameters of the atmosphere, and to construct mathematical models for simulating these random components with analog and digital computers as applied to the problems of control flight vehicle flight. Great attention in the book is given to generalizing and systematizing methods of the statistical investigation of the control of flight vehicle motion in the dense atmospheric layers, formulating methods of evaluating the effect components of the /5 physical parameters of the atmosphere have on the control process, and presenting methods of the numerical optimization of flight vehicle control systems.

The exposition of the material is addressed to the reader with a mathematical background at the level of the higher educational institution and familiar with the fundamentals of probability theory, the theory of random functions, and automatic control theory.

We deem it our pleasant duty to express our deep gratitude to professors M. I. Yudin and Ye. P. Borisenkov, Candidate of Physico-mathematical Sciences A. I. Ivanovskiy, and Candidate of Technical Sciences A. A. Lukashevskiy, who at various stages of the preparation of the manuscript made a number of valuable comments that improved the book.

The authors will be appreciative of readers who will find it possible to send their critical remarks, responses, and wishes to the Press.

INTRODUCTION

/6

In addition to the thrust force, gravity force, and aerodynamic forces, the motion of flight vehicles (aircraft, helicopters, rockets, spacecraft, and other moving objects) in the earth's atmosphere is also affected by a large number of random actions. They usually include fluctuations in engine thrust, forces associated with random skewing in the installation of wings, stabilizers, combustion chambers, and engine frame; random components of aerodynamic forces, and so on. To a large extent, the motion of flight vehicles is determined by the condition of the earth's atmosphere.

The condition of the earth's atmosphere is characterized by a number of physical parameters, which undergo extensive variability under the effect of processes occurring within the atmosphere itself (cyclonic and anticyclonic activity, convective and turbulent heat currents, and so on), and also under the effect of processes occurring in the Sun (fluxes of electromagnetic and corpuscular radiation). The state of the atmosphere is also determined by the time of the year, time of the day, and latitude of the location.

These processes determine principally the condition of the dense atmospheric layers, by which we mean the lower layers of the atmosphere (troposphere, stratosphere, and mesosphere).

When a flight vehicle is moving in the dense atmospheric layers, it is acted on by thrust P, force of gravity G, and aerodynamic forces Q, Y, and Z. Let us consider equations of the motion of the mass center of a flight vehicle in the terrestrial coordinate system. The origin of coordinates O_1 of this system is fixed relative to the earth, the O_1y axis is directed along the force of gravity G, the O_1x axis is directed along flight vehicle motion, and the O_1z axis is perpendicular to the O_1x and O_1y axes and is directed so as to constitute a right-handed coordinate system.

Since the aerodynamic forces depend on the direction of the velocity vector of the flight vehicle mass center, we introduce a wind-axes coordinate system. The origin of coordinates of this system is in the craft mass center. The Ox_W axis is oriented along the velocity vector, the Oy_W axis is perpendicular to the Ox_W axis and lies in the plane of longitudinal flight vehicle symmetry, and the Oz_W axis is oriented perpendicular to the Ox_W and Oy_W axes in the right wing of the flight vehicle when it in forward motion.

Motion of the individual parts of the flight vehicle with respect to its mass center is determined in a body-axes coordinate system, whose origin of coordinates also lies in its mass center. Here the Ox_b axis is directed parallel to the longitudinal flight vehicle axis, the Oy_b axis is perpendicular to Ox_b and lies in the plane of longitudinal symmetry, and the Oz_b axis is directed along the right wing of the flight vehicle perpendicular to Ox_b and Oy_b. /7

Equations of force equilibrium in projections onto the axis of the wind-axes coordinate system **are of the form:**

$$\left.\begin{aligned} m\frac{dv}{dt} &= P\cos\alpha\cos\beta - Q - G\sin\theta; \\ mv\frac{d\theta}{dt} &= P\sin\alpha + Y - G\cos\theta; \\ \cos\theta\, mv\frac{d\psi_W}{dt} &= P\cos\alpha\sin\beta - Z. \end{aligned}\right\} \qquad (1)$$

Here m is flight vehicle mass; v is flight vehicle velocity; θ is the angle of inclination of velocity vector to the horizon; ψ_W is the wind-axes yaw angle (angle between the projection of velocity vector onto the plane of the horizon and the O_1x axis); α is the angle of attack (angle between projection of flight vehicle velocity vector onto the longitudinal plane of symmetry y_bx_b and the longitudinal axis); and β is the slip angle (angle between velocity vector and its projection onto the longitudinal plane of symmetry);

$$\left.\begin{aligned} Q &= C_x\frac{\rho v_a^2}{2}S, \\ Y &= C_y\frac{\rho v_a^2}{2}S, \\ Z &= C_z\frac{\rho v_a^2}{2}S \end{aligned}\right\} \qquad (2)$$

are the projections of the aerodynamic forces on the axes of the body-axes coordinate system; C_x, C_y, and C_z are the aerodynamic coefficients; $q = \rho v_a^2/2$ is the velocity head; v_a is air speed of the flight vehicle; ρ is density of air; and S is the characteristic dimension to which the coefficients are referred.

Air speed v_a is determined by the velocity of the flight vehicle and the projections of wind velocity along the meridian and parallel (u and v), and also by the projection of wind velocity w along the vertical.

The equations of motion of the flight vehicle mass center can be written as: /8

$$\left.\begin{aligned} \frac{dx}{dt} &= v\cos\theta\cos\psi_W; \\ \frac{dy}{dt} &= v\sin\theta; \\ \frac{dz}{dt} &= v\cos\theta\sin\psi_W \end{aligned}\right\} \tag{3}$$

The systems of equations (1) and (3) describe the motion of the flight vehicle mass center in the earth-based coordinate system. From Eqs. (2) it follows that the aerodynamic forces are determined by the density of air ρ. The aerodynamic coefficients C_x, C_y, and C_z, for near-sonic, sonic, and supersonic flight vehicle velocities, depend on the angle of attack α and slip angle β, M number, and the Reynolds number Re:

$$\left.\begin{aligned} C_x &= C_x(\alpha, \beta, M, \mathrm{Re}); \\ C_y &= C_y(\alpha, \beta, M, \mathrm{Re}); \\ C_z &= C_z(\alpha, \beta, M, \mathrm{Re}); \\ M &= \frac{v}{a};\ \ \mathrm{Re} = \frac{\rho v_a l}{\nu}, \end{aligned}\right\} \tag{4}$$

where l is the length of the flight vehicle, ν is the dynamic viscosity of air, and v_a is the speed of sound in air.

The speed of sound is determined in terms of the characteristic of the unperturbed flow by the formula

$$a = \sqrt{kRT},$$

where k is the ratio of specific heat capacities; R is the gas constant of air, and T is the absolute temperature of the unperturbed flow.

Summing up the foregoing and considering the relationship between temperature, pressure, and density of air, it can be noted that the motion of a flight vehicle in dense atmospheric layers is determined by the thermodynamic parameters of the air (density, pressure, and temperature) and by the wind.

When calculating the trajectories of flight vehicles in dense atmospheric layers, use is made of the standard atmosphere (the SA-64 standard atmosphere is adopted in the USSR for altitudes to +200,000 m).

However, actual trajectories as a rule differ widely from calculated trajectories. One reason for this is the deviations in the actual state of atmospheric parameters (T, p, and ρ) from the values (T_{st}, p_{st}, ρ_{st}) adopted in the model:

$$\Delta T = T - T_{st},$$
$$\Delta p = p - p_{st},$$
$$\Delta \rho = \rho - \rho_{st}.$$

There are three directions which can be pursued in allowing for the effect of the atmosphere on flight vehicle motion. The first of these is to use the actual distribution of the physical parameters of the atmosphere. Essentially it is the most effective approach, however at the present time there are not yet available methods of determining the state of the atmospheric parameters with the required precision and completeness for a specified time. /9

The second direction involves determining the values of the physical parameters of the atmosphere by using hydrodynamic models. Unfortunately, these models have not yet been developed for the stratosphere and mesosphere owing to the very great difficulties of mathematically describing the processes occurring therein, while existing hydrodynamic models capable of precalculating the temperature and pressure fields in the troposphere do not thus far provide the required precision.

Finally, the third direction proposes using the statistical characteristics of the physical parameters of the atmosphere; essentially it amounts to the following. As we know, the motion of a flight vehicle is described in the general case by a system of nonlinear differential equations, whose right sides include the external random perturbations, including atmospheric perturbations. If the statistical characteristics of the atmospheric perturbations are known, various methods of statistical analysis of dynamic systems make it possible to determine the characteristics of the scatter of flight vehicle trajectories in the dense atmospheric layers.

The foregoing applies fully to problems of controlling space flights. In this case the statistical characteristics of fluctuations in the physical parameters of the atmosphere are used in solving the problems of optimizing flight control systems and predicting the state of the atmosphere for control purposes. In this monograph, the authors place their emphasis precisely on this third direction.

The monograph consists of two parts that are interrelated, but at the same time have a certain autonomy. This autonomy lies in the fact that **results** presented in the two parts of the mongraph can be used in solving other problem unrelated to the particular problem merely under study here.

~~The first part includes~~ the first four chapters of the book dealing with the atmosphere proper. Chapter One examines the temperature regime and the distribution of pressure and density of air in the dense atmospheric layer, briefly analyzes the causes responsible for particular changes in these physical parameters of the atmosphere, and indicates the limits of possible variations in these parameters. The wind regime in the troposphere, stratosphere, and mesosphere is the subject of the second chapter. Chapter Three contains information on the vertical statistical structure of the temperature, pressure, air density, and wind fields in the dense atmospheric layers. The numerous statistical /10
characteristics given in this chapter were obtained for two groups of latitudes based on data of American rocket sounding. They are analyzed in detail and evaluated from the standpoint of their statistical significance.

Chapter Four is a bridge connecting the structure of fields of physical parameters of the atmosphere with questions of the statistical analysis of dynamic systems and the synthesis of controls. It examines several statistical models of the physical parameters of the atmosphere. The theoretical conclusions contained in the chapter are extensively illustrated with specific examples of models of temperature, air density, and wind velocity components.

The second part of the book consists of the four last chapters and expounds methods of the statistical investigation of the control of flight vehicle motion in the dense atmospheric layers described by nonlinear differential equations.

In Chapter Five, based on the above-indicated statistical models, mathematical models of the motion of flight craft in dense atmospheric layers described by nonlinear differential equations are constructed. Primary attention in this chapter is given to methods of the statistical analysis of the scatter of parameters of flight vehicle trajectories when various mathematical models are used. In addition to methods of the statistical investigation of nonlinear processes based on linear approximation, approximate numerical methods of the statistical analysis of nonlinear systems are set forth (the method of statistical tests, the B. G. Dostupov method, the interpolation method, and so on); ways of using the method of least squares in statistical analysis are outlined. Urgent problems in the investigation of flight vehicle motion in the earth atmosphere include evaluation of the effect of atmospheric perturbations on the scatter of trajectories. Chapter Six in fact deals with methods of solving these problems. Much attention in it is given to setting forth a method of stochastic approximation as applied to problems of setting up polynomial dependences of motion parameters on random factors characterizing atmospheric perturbations.

Chapter Seven sets forth numerical methods of the statistical optimization of control processes of flight craft motion in dense atmospheric layers. Elaborating numerical methods of the statistical optimization of control processes became possible thanks to the use in control system design practice of high-speed digital computers. This then determined the nature of the methods examined in the book and their algorithmic structure. This chapter gives a comparative characterization of a number of computational prospects of methods of searching for the extremum for the statistical characteristics of stochastic control processes.

The methods and algorithms of statistical optimization considered in this chapter can be successfully employed also in optimizing stochastic processes of controlling various objects of other types described by nonlinear stochastic differential equations. /11

Chapter Eight deals with methods of statistical prediction of the parameters of flight vehicle motion described by nonlinear stochastic differential equations. Mathematical relations for solving problems of predicting phase coordinates are derived within the framework of regression analysis.

The book presents a large number of statistical characteristics of the physical parameters of the atmosphere and the results of solving practical problems.

TABLE OF CONTENTS

	Page
Author's Abstract	iii
Editor's Foreword	iv
Authors' Foreword	v
Introduction	vii
Table of Contents	xiii
CHAPTER ONE. DISTRIBUTION OF TEMPERATURE, PRESSURE, AND AIR DENSITY IN THE DENSE ATMOSPHERIC LAYERS	1
1.1. Temperature Regime in the Troposphere, Stratosphere, and Mesosphere	1
1.2. Distribution of Pressure in the Troposphere, Stratosphere, and Mesosphere	9
1.3. Density of Air in the Troposphere, Stratosphere, and Mesosphere	12
1.4. Abrupt Warming in the Stratosphere and the Associated Variations in Thermodynamic Parameters of the Atmosphere	15
CHAPTER TWO. STRUCTURE OF WIND FIELD IN THE TROPOSPHERE, STRATOSPHERE, AND MESOSPHERE	20
2.1. Wind Regime in the Troposphere, Stratosphere, and Lower Mesosphere	20
2.2. Distribution of Wind in the Layer of Meteor Trails	24
2.3. Turbulent Motions in Dense Atmospheric Layers	26
CHAPTER THREE. VERTICAL STATISTICAL STRUCTURE OF PHYSICAL PARAMETERS OF AIR IN DENSE ATMOSPHERIC LAYERS	42
3.1. Characteristics of Vertical Statistical Structure of Physical Parameters of the Atmosphere	42
3.2. Altitude Distribution of Mean Values of Temperature, Pressure, and Air Density in the Dense Atmospheric Layers	48
3.3. Correlation Matrices of Temperature, Pressure, and Air Density	61
3.4. Statistical Characteristics of Wind in Dense Atmospheric Layers	71
3.5. Precision of Determining the Statistical Characteristics of the Physical Parameters of the Atmosphere	77

CHAPTER FOUR. FORMS OF REPRESENTATION OF ATMOSPHERIC PERTURBATIONS 84
4.1. Approximation of Autocorrelation Functions of Air Temperature and Density and of Wind with Analytic Expressions 84
4.2. Noncanonical Expansions of Temperature and Air Density and Wind Velocity Components 98
4.3. Canonical Expansions of Air Temperature and Density and of Wind Velocity Components 103
4.4. Shaping Filters of Physical Parameters of the Atmosphere 116
4.5. Representation of Atmospheric Perturbations Using Eigenelements of the Correlation Matrices of the Physical Parameters of the Atmosphere 120
4.6. Discrete-continuous Representation of Random Processes Within the Frame of Correlation Theory 131

CHAPTER FIVE. STATISTICAL METHODS OF INVESTIGATING THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS 138
5.1. Models of Nonlinear Control Processes 138
5.2. Statistical Analysis of Nonlinear Systems Based on a Linear Approximation 157
5.3. Methods of Statistical Investigation of Nonlinear Processes 168
5.4. Method of Statistical Analysis of Control Processes with Discrete-Continuous Model of Perturbations 192

CHAPTER SIX. INVESTIGATING THE EFFECT OF ATMOSPHERIC PERTURBATIONS ON THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS 195
6.1. Investigating the Significance of Atmospheric Perturbations 195
6.2. Method of Stochastic Approximation 201
6.3. Constructing Polynomials of First, Second, and Third Degrees in Stochastic Experiment Planning 210
6.4. Precision of Approximation in Stochastic Experiment Planning 220
6.5. Computation of Estimates of Moments Z 222

CHAPTER SEVEN. NUMERICAL OPTIMIZATION OF CONTROL ALGORITHMS FOR FLIGHT VEHICLE MOTION 226
7.1. Optimization of Processes of Controlling Flight Vehicle Motion 226
7.2. Methods of Searching for Extremum of Functions of Many Variables 229
7.3. Methods of Computing Partial Derivatives for Statistical Characteristics of Stochastic Processes 239

7.4. Methods of Constructing Approximating Polynomials 252
7.5. Method of Stochastic Approximation in Problems of Constructing Approximating Polynomials 254
7.6. Methods of Allowing for Constraints in Problems of Optimizing Control Systems 266

CHAPTER EIGHT. STATISTICAL PREDICTION PROBLEMS OF CONTROLLING THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS 276
8.1. Control of Motion of Flight Vehicles in the Atmosphere with the Prediction of Phase Coordinates 276
8.2. Predicting Phase Coordinates of Nonlinear Stochastic Processes 278
8.3. Determination of the Effectiveness of Control Actions on the State of the Process 292
8.4. Optimal Controls with Prediction of Phase Coordinates 297

REFERENCES 301

THE ATMOSPHERE AND THE CONTROL OF FLIGHT VEHICLE MOTION

Ye. P. Shkol'nyy and L. A. Mayboroda

edited by Doctor of Technical Sciences,
Professor V. M. Ponomarev

CHAPTER ONE /12

DISTRIBUTION OF TEMPERATURE, PRESSURE, AND AIR DENSITY IN THE DENSE ATMOSPHERIC LAYERS

1.1. Temperature Regime in the Troposphere, Stratosphere, and Mesosphere

The temperature regime is one of the main factors determining the physical state of the atmosphere. The nonuniform distribution of temperature in the atmosphere accounts for the specific structure of the pressure field and, therefore, of atmospheric circulation relative to the earth's surface.

Change in temperature in the atmosphere occurs under the action of two main causes. The first is the interaction of the atmosphere with the underlying surface, and the second is represented by processes occurring within the atmosphere itself.

Change in temperature with time at some point in space can be described with the equation

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_p}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \frac{A}{\rho c_p}\frac{dp}{dt} - \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - w\frac{\partial T}{\partial z}, \quad (1.1)$$

in which ρ is the density of air; c_p is the heat capacity of air at constant pressure; A is the thermal equivalent of work; ϵ_1, ϵ_2, and ϵ_3 are influxes of heat caused by the turbulent thermal conductivity, radiative heat transfer, and phase transformations of water in the atmosphere, respectively; u, v, and w are the components of their velocity relative to the x, y, and z axes (the x axis is tangent to the circles of the latitudes, the y axis is tangent to the meridians, and the z axis is directed vertically upward); T is air temperature in an absolute scale; and p is air pressure.

In the troposphere, the second term in Eq. (1.1), can be neglected owing to its smallness. In the upper atmosphere, as shown in the period of atmospheric tides, this term now plays an essential role. The remaining terms in Eq. (1.1) describe temperature change due to the influx of heat, advective transport, and adiabatic ascent or descent of air masses.

The main source of heat for the earth's surface and the atmophere is solar radiation. The radiant energy of the Sun, on passing through the atmosphere, is considerably weakened. Its weakening occurs due to scattering and absorption by molecules and atoms of the gases comprising the atmosphere and also by impurities present in the air. However, most of the solar energy penetrates through the atmosphere and is absorbed by the earth's surface. In turn, the surface of the earth is a source of long-wave radiation, which is absorbed by the atmosphere. The difference between the amount of absorbed direct and scattered radiation and the radiation by the underlying surface that is, the radiation budget of the earth's surface is one of the most important factors determining the temperature regime of the lower atmospheric layer -- the troposphere. Investigations showed /55/ that in the equatorial zone between 39°N. and S. Lat the radiation budget is positive throughout the year. To the north and to the south of this zone it is negative in the cold period of the year. /13

The transfer of heat from the underlying surface to the air takes place by a turbulent exchange and long-wave radiation. Of high importance in the heat regime of the troposphere is the heat realized from the phase transformations of water. Turbulent transport plays a basic role in the lower troposphere. In the upper troposphere it markedly weakens and radiative heat flux becomes determining. In the stratosphere, as shown by studies /18/, turbulent thermal diffusivity plays an essential role. In any case, without allowing for turbulent heat transport, we are unable to understand how the radiation conditions existing in the stratosphere lead to the formation of temperature profiles. Evidently, this is true also of the mesosphere. The main absorbing components of the atmosphere are water vapor, carbon dioxide gas, and ozone.

The amount of heat which arrives per unit area of the earth's surface in the lower latitudes during the year considerably exceeds the amount of heat arriving in the upper latitudes. Thus, the heating of the earth's surface decreases on the average from equator to pole. Accordingly, a horizontal temperature gradient is induced in the troposphere oriented from pole to equator.

While the main heat source for the troposphere is the earth's surface heated by solar rays, in the stratosphere and mesosphere the distribution of temperature by altitude and latitude as well as its seasonal changes are determined by the absorption of short-wave solar radiation and also by radiation of the troposphere in the infrared spectral region. Ozone is most significant in the absorption of ultraviolet solar radiation in the stratosphere and mesosphere.

Ozone is observed in the atmospheric layer from the earth's surface to an altitude of 70-80 km, but most of it is concentrated at altitudes of 20-25 km. The Hartley band (1800-3400 A) plays the principal role in the absorption by ozone of the ultraviolet spectral region. Absorption of this spectral region of solar radiation leads to heating of the atmosphere; its maximum occurs at altitudes of 45-55 km. In addition, atmospheric heating also occurs owing to the absorption of solar radiation by molecular oxygen. It is evident mostly noticeably at altitudes higher than 90 km. The radiation of heat in the stratosphere and mesosphere occurs principally in the 15 μm band for carbon dioxide gas and the 9.6 μm band for ozone. The overall effect of heating and cooling, as shown by investigation /26/ does not exceed 1°/day for the summer season in the 14-30 km atmospheric layer. Thus, this layer is close to the state of radiative equilibrium. In winter it somewhat increases from equator to polar latitudes. /14

The upper stratosphere and the lower mesosphere are a strong heat source in summer, with its maximum intensity in the polar region of the summer hemisphere. In the winter hemisphere, with the exception of the polar latitudes, heating also predominates over cooling, however here the heat sources are weaker than in the summer hemisphere. In the polar latitudes in winter, in contrast, cooling predominates over heating. In general, this pattern is observed also in the upper mesosphere, though heat sources and sinks in it are weaker. In the region of the mesopause, for example, a small heat influx predominates; its maximum, 4°/day, lies near the summer pole.

Studies showed /109/ that the stratosphere and the mesosphere are highly sensitive to fluctuations in solar radiation. A 12% change in the absorption of ultraviolet radiation by ozone above 35 km, and by oxygen above 20 km leads to a 2° temperature change. The stratosphere in the 20-35 km layer is even more sensitive to fluctuations in the absorption of solar radiation in the visible spectral region. For example, a 3-6% rise in the visible radiation also leads to a $2^{\circ}C$ rise in temperature in the atmospheric layer indicated above. Therefore the reflection of visible radiation downward, especially from clouds, is highly significant. It is established that large and dense cloud fields observed in the middle latitudes in winter can increase through reflection the amount of radiation arriving at the stratosphere in the visible spectral region by 35%, which corresponds to roughly a 10% rise in temperature in some of its region /109/.

Radiation conditions in the stratosphere and mesosphere lead to the horizontal temperature gradient in summer being directed south to north, and in winter -- north to south (here and in the following we consider the gradient as a vector directed toward the side of increasing function). A consequence of this temperature distribution is that in winter westerly transport of air masses develops in these atmospheric layers. In summer, in contrast, transport from east to west predominates. In the troposphere however westerly transport is predominant in all seasons of the year. However, in the zonal circulation waves continually are generated which lead to the appearance of the meridional component /15 of air motion. Advective transport of cold air masses to the south occurs, and of warm air masses to the north. As a result of the interlatitudinal exchange, redistribution of energy between the warm and cold atmospheric regions is observed. Interlatitudinal exchange acquires its highest intensity during the cold half of the year.

Advective temperature changes in the troposphere can sometimes exceed 10°C per day. Advective temperature changes of similar magnitude are observed in the stratospheres as well. Above 40 km, advective transport of air masses can lead to even greater changes in air temperature. In the stratosphere over the White Sands station, in a day the temperature can rise 10°C and higher and drop 30°C or more owing to the advective factor in the 40-km atmospheric layer [88]. Advective temperature changes of similar magnitude in the atmosphere have been obtained also in the studies [63, 106].

As indicated above, adiabatic temperature changes also occur in the atmosphere; they are associated with ordered large-scale vertical motions. In the troposphere ascending and descending motions result from thermal convection and horizontal convergence or divergence of air currents arising owing to turbulent friction in the boundary layer of the atmosphere and the nonsteady state of processes occurring in the atmosphere. The most important role in the thermal regime of the atmosphere is played by ordered vertical motions embracing quite large air volumes. The velocity of ordered vertical motions is low. It averages 1-2 cm/sec and can vary from several tenths to 5-10 cm/sec. They lead to a 3-5°C temperature change per day in the troposphere.

In the stratosphere and mesosphere, ordered vertical motions also are small in summer [94, 106]. Their velocity as a rule is several mm/sec. In winter vertical motions in the stratosphere have approximately the same value as in the troposphere. Vertical motions in this atmospheric layer also lead to temperature change at the corresponding levels, where these changes are the greater, the more stable is the temperature stratification of the air.

Based on data obtained from direct and indirect observations, V. R. Dubentsov [16] constructed a vertical meridional profile of temperature up to the altitude of 100 km (Fig. 1.1). In the profile is shown the position of the transition of layers between the spheres at which the lower 100-kilometer atmospheric layer

is divided. The temperature field, as shown in Fig. 1.1, has several features. In the stratosphere and lower mesosphere there is situated in summer a region of heat with maximum temperatures over the polar region. A further rise in altitude is accompanied by a rapid drop in temperature, and in the upper mesosphere there is now found a region of cold, where the lowest temperatures are /16
also observed in the polar region.

In winter, the temperature in the lower stratosphere decreases to an altitude of approximately 30 km. Above 30 km it somewhat rises and then again falls. The vertical temperature gradients in winter in the mesosphere have a considerably smaller value than in summer.

Data from rocket sounding of the atmosphere showed that in the stratosphere of the Arctic there is a nearly isothermal atmospheric layer from 10 to 30 km /78/. Only above 30 km is there a rise in air temperature with altitude. In the temperate latitudes the isothermal layer is somewhat narrower. The vertical /17
temperature gradient at the altitudes 50-55 km in the polar latitudes changes over relatively wide limits. Whereas at the outset of the polar night it averages 1.5°/km, by the end of the polar night it is 5.5°/km. In the middle latitudes, the vertical temperature gradient in this atmospheric layer averages about 2°/km.

The vertical temperature profile in the tropical latitudes can be judged only from episodic observations taken with meteorological rockets.

Investigations showed that in the equatorial zone seasonal temperature changes in the stratosphere, mesosphere, and lower thermosphere are small /114/. Small seasonal changes in air temperature in the lower latitudes is also indicated in /89/. The authors of /89/, subjecting the air temperature extrapolated to 15° N. Lat to harmonic analysis, obtained the amplitudes and phases of the annual and semiannual cycles of temperature change at different altitudes (Table 1.1).

As follows from Table 1.1, the amplitude of annual and semiannual temperature fluctuations in the atmospheric layer from 37.5 to 52.5 km is approximately identical. At the lower levels, the amplitude of semiannual fluctuations is nearly three times greater than the amplitude of the annual fluctuations.

Table 1.2 presents annual differences between the maximum and minimum mean-monthly temperatures for 15, 30, and 60° N. Lat by Cole, Kantor, and Nee /89/, and for 40, 50, 70, and 80°N. Lat by L. A. Ryazanova /63/.

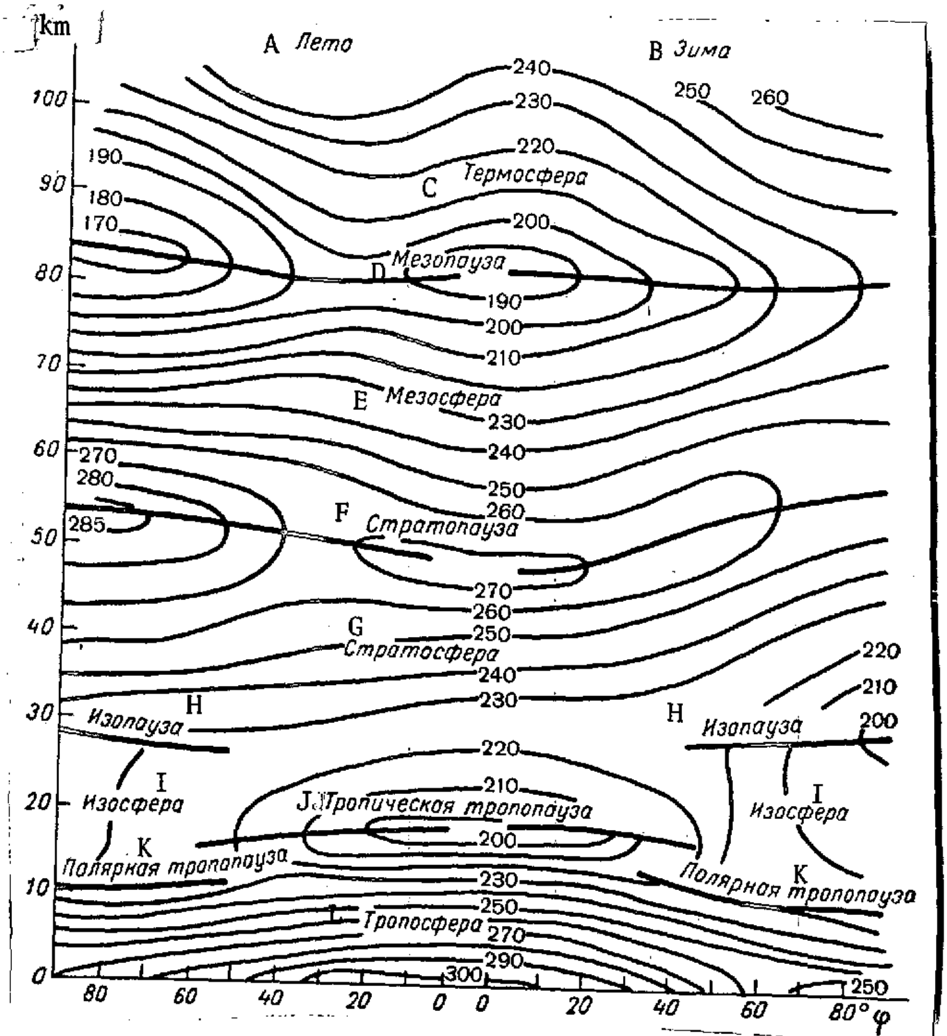


Fig. 1.1. Distribution of temperature to altitude of 100 km

KEY:
- A. Summer
- B. Winter
- C. Thermosphere
- D. Mesopause
- E. Mesosphere
- F. Stratopause
- G. Stratosphere
- H. Isopause
- I. Isosphere
- J. Tropical tropopause
- K. Polar tropopause
- L. Troposphere

TABLE 1.1. AMPLITUDES AND PHASES OF ANNUAL AND SEMIANNUAL CYCLES OF TEMPERATURE CHANGE FOR THE TROPICAL ZONE (15° N. Lat)

Altitude km	6-month cycle		12-month cycle	
	amplitude, deg.	date of 1st max.	amplitude, deg.	date of 1st maximum
27.5	2.2	21/IV	0.8	12/V
32.5	2.8	3/V	0.8	21/I
37.5	1.2	9/IV	1.3	3/IV
42.5	1.1	24/III	1.4	27/I
47.5	1.3	27/III	1.2	3/I
52.5	2.0	1/III	1.6	6/XII

From Table 1.2 it follows that the amplitude of annual fluctuations in temperature rises with latitude, and especially rapidly at latitudes higher than 40°, where its increase is by a factor of 3-4.

TABLE 1.2. ANNUAL DIFFERENCES BETWEEN MAXIMUM AND MINIMUM MEAN MONTHLY TEMPERATURES (°C) AT DIFFERENT ALTITUDES AND LATITUDES /18

Altitude km	Latitude, degrees						
	15	30	40	50	60	70	80
25.0		4.9	11.0	23.0	18.0	34.0	40.0
27.5	5.4	5.5			20.2		
30.0		7.7	11.0	21.0	23.0	37.0	46.0
32.5	6.5	8.2			24.0		
35.0		8.6	9.0	22.0	28.0	40.0	38.0
37.5	3.9	7.9			28.8		
40.0		7.2	8.0	28.0	29.5	39.0	42.0
42.5	4.2	7.0			30.3		
45.0		6.7		22.0	27.2	40.0	36.0
47.5	3.4	5.4			23.2		
50.0		5.0			17.0		
52.5	7.0	4.0			17.0		

The amplitude of the annual fluctuation is small. Table 1.3 contains the amplitudes of diurnal and semidiurnal temperature fluctuations over the Azores /99/ obtained by harmonic analysis of the data of atmospheric radio sounding.

TABLE 1.3. AMPLITUDE OF DIURNAL AND SEMI-DIURNAL CHANGES IN TEMPERATURE OVER THE AZORES

Altitude, km	Amplitude, °C		Altitude, km	Amplitude, °C	
	semidiur	diurnal		Semidiur	Diurnal
1.5	0.02	0.20	12.0	0.04	0.29
3.0	0.02	0.14	16.0	0.09	0.40
5.5	0.04	0.19	20.0	0.16	0.68
9.0	0.04	0.27	24.0	0.09	0.78

The amplitude of the diurnal trend of temperature rises to 1.0-1.5°C at the altitudes 25-30 km, and to 3-5°C at the altitudes 40 km and higher /7/.

Special investigations of the diurnal trend of temperature in the atmospheric layer from 30 to 60 km were conducted over the American White Sands Proving Grounds /88/. For this purpose, 11 meteorological rockets were launched with 2-hour intervals between launches.

Results of the observations are in Fig. 1.2, from which it follows that the diurnal trend of temperature is best defined at the altitudes 45-55 km. The temperature maximum in this atmospheric layer is observed at the instant of time close to 14:00, and the temperature minimum -- about 4:00 local time. The difference between the maximum and the minimum temperature at these altitudes was 15-20°C. Below 45 km, the diurnal temperature trend is less strongly pronounced. /19

A great deal of attention recently is being given to investigating correlations between air temperature at different levels in the troposphere and lower stratosphere. It was shown /30, 38, 68, 69/ that in the troposphere of the high and middle latitudes, the autocorrelation of temperature with increase in altitude decreases. At the level of the troposphere the coefficients of correlation decrease to zero and above this level they become negative. The root mean square deviations of temperature from a limited series of rocket sounding data (about 200 domestic and foreign soundings) were obtained by A. S. Borovikova and O. B. Mertsalova /8/, and also by V. G. Kidiyarova /27/ for the 30-80 km atmospheric layer. They indicate that temperature variability in this atmospheric layer is 6-12%. Statistical characteristics of temperature and atmospheric parameters will be discussed in greater detail below.

1.2. Distribution of Pressure in the Troposphere, Stratosphere, and Mesosphere

General correlations of change in pressure with altitude are simpler than for change in air temperature with altitude. While air temperature in some atmospheric layers falls off, and rises in others, pressure steadily decreases with altitude. However, the rate of the pressure decrease is not the same. It depends on air density. In the lower atmospheric layers where the air density is higher, pressure falls off faster, and in the upper layers -- more slowly. In addition, if one considers that the air density depends not only on altitude above sea level, but also on /20 air temperature, it can be concluded that pressure at the same altitudes can vary in different ways.

Two factors affect the baric relief of the atmosphere at a specific altitude: pressure at sea level and the mean temperature of the layer enclosed between sea level and the surface under consideration. The effect of these factors is not the same everywhere. With increasing altitude, the mean temperature of the layer plays an ever larger role, and sea level pressure -- an ever smaller role. Even in the middle troposphere as a rule nearly complete correspondence between the lower values of the altitudes of isobaric surfaces with the lower values of the mean temperatures is observed. Therefore, altitude regions of reduced pressure (cyclones and troughs) are formed where there is relatively cold air mass, and the altitude regions of increased atmospheric pressure (anticyclones and ridges) are formed in warm air masses. In a few words, the structure of the pressure field in the atmosphere depends on the structure of the temperature field.

As noted in Section 1.1, in the troposphere the highest air temperatures are observed in the tropical zone, and the lowest -- in the polar regions. This distribution of heat and cold is characteristic both of summer and winter, therefore isobaric surfaces in the troposphere on an average are situated the highest in the tropics /55/.

In the stratosphere, the baric field, like the temperature /21 field, has a well-pronounced seasonality. Since in summer the polar region heats up most strongly and, therefore, the temperature gradient is oriented south to north, the highest values of the geopotential are observed over the pole. The isobaric surfaces become lower toward the south. A baric field typical for the summer season is shown in Fig. 1.3 /24/. In winter, in contrast, the lowest values of the altitudes of isobaric surfaces in the stratosphere are noted over the pole, as for example, in the map of the baric topography of the 10 mb isobaric surface on 1 January 1962 (Fig. 1.4). Thus, in winter the baric field in the stratosphere has a similarity with the baric field in the troposphere.

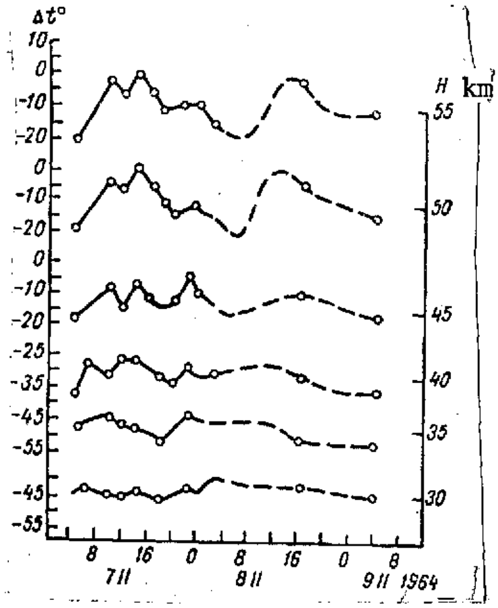


Fig. 1.2. Diurnal trend of temperature over White Sands

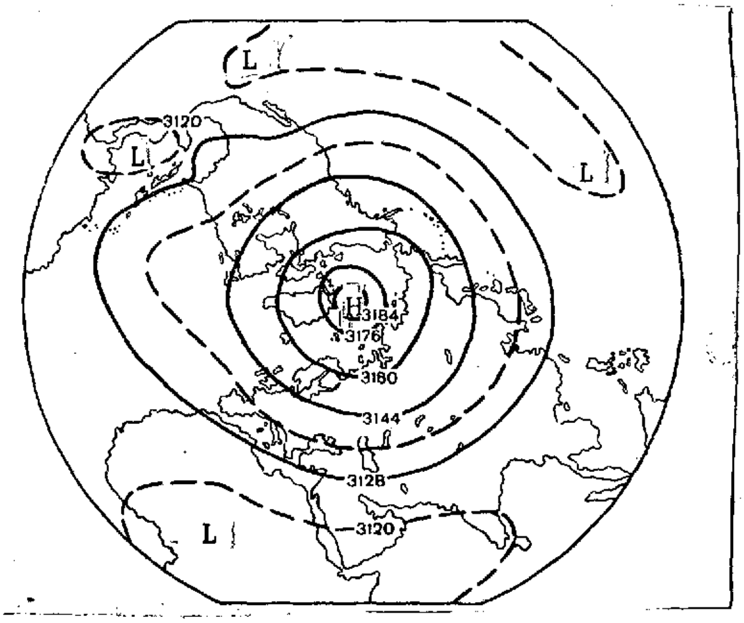


Fig. 1.3. Map of altitudes of 10 mb isobaric surface on 23 May 1959

Investigations showed that seasonal variations in the mean pressure values in the stratosphere and mesosphere have large values, especially in the polar regions /27/. In winter they are identical throughout the entire atmospheric layer from 30 to 70 km and average 20%; in summer the increase with altitude from 20% at 30 km altitude to 32% at 50 km altitude; they remain nearly constant with further increase in altitude.

Variability in atmospheric pressure relative to a seasonal mean proved to be greater in winter than in summer /27/. Whereas in winter the pressure fluctuates in the range 25-35% in the 30-80 km layer, in summer its variability is 10-15%.

In addition to seasonal changes, atmospheric pressure exhibits diurnal variations (Table 1.4). /22

Table 1.4 contains diurnal and semidiurnal variations of pressure over the Azores /99/.

From Table 1.4 it follows that up to an altitude of 5.5 km, the amplitude of the semidiurnal changes exceeds the amplitude of the diurnal pressure changes. The amplitude of the semidiurnal variations rapidly decreases with altitude and becomes insignificant in the middle stratosphere. A further increase in altitude is related to a rise in semidiurnal pressure variations. The amplitude of diurnal pressure changes reaches a maximum at altitudes

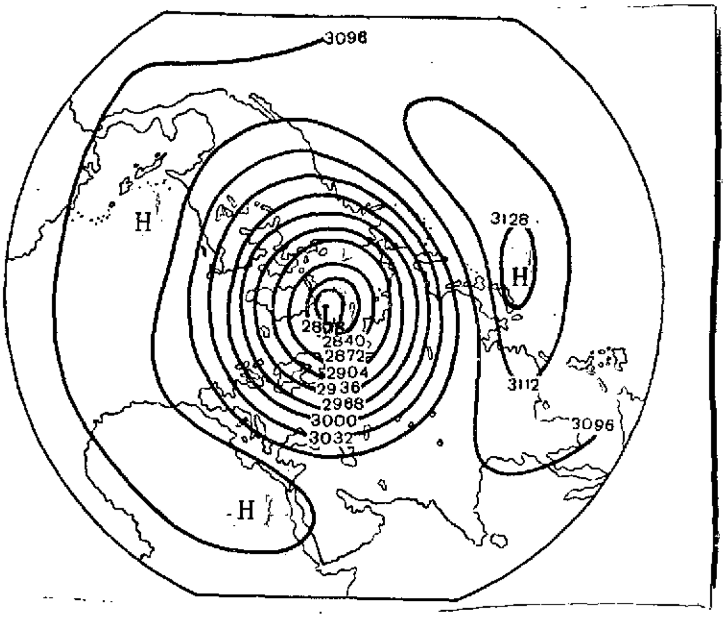


Fig. 1.4. Map of altitudes of 10 mb isobaric surface on 1 January 1962

9-12 km and thereupon decreases slowly with altitude. But the relative amplitudes in diurnal variations of pressure increases with altitude. Whereas at an altitude 3 km the annual mean amplitude of diurnal pressure changes is 0.03%, at the altitude 9 km it rises to 0.1% at the altitude 20 km -- to 0.4%, and at the altitude 30 km -- to 1.2%. Diurnal pressure changes increase with altitude even above 30 km, especially in the high latitudes in winter.

Data of atmospheric rocket sounding showed that in a day the pressure

TABLE 1.4. AMPLITUDES OF DIURNAL AND SEMIDIURNAL VARIATIONS IN ATMOSPHERIC PRESSURE OVER THE AZORES [99]

Altitude, km	Amplitude, mb		Altitude, km	Amplitude, mg	
	diurnal	semidiur		diurnal	semidiurnal
1.5	0.18	0.47	16.0	0.26	0.09
3.0	0.25	0.37	20.0	0.21	0.07
5.5	0.28	0.29	22.0	0.18	0.05
9.0	0.32	0.18	30.0	0.12	0.01
12.0	0.32	0.14			

in the stratosphere of the upper latitudes can vary 10-15%. For example, over the settlement of Fort Greeley the air pressure at the altitude 35 km decreased 17% from 10 to 11 March 1964, and increased 13% at the altitude 45 km from 26 to 27 October.

1.3. Density of Air in the Troposphere, Stratosphere, and Mesosphere

The pressure distribution in the dense atmospheric layers depends on the structure of the temperature field, therefore the air pressure is a function of the mean temperature of the underlying atmospheric layer. The air density in some point in space, in accordance with the equation of state, is determined by temperature and pressure.

Air pressure falls off rapidly with altitude. But changes in air pressure in the dense atmospheric layers are quite large at /23 all altitudes. The air temperature in an absolute scale varies relatively little. This circumstance is responsible for the fact that a change in air density with increase in altitude is increasingly determined by pressure change. This dependence of air density on temperature and pressure shows up well if one compares the distribution of mean pressure and mean air density with altitude.

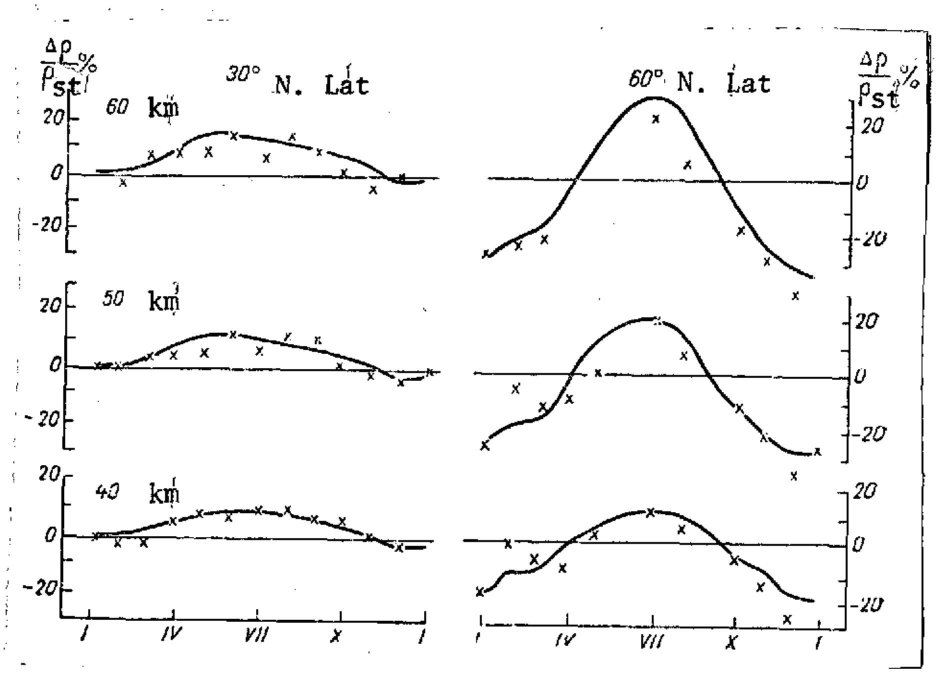


Fig. 1.5. Annual trend of mean monthly values of air density in deviations from the 1952 US Standard Atmosphere

A study [104] showed that the highest air density is observed in summer, and the lowest in winter. The maximum seasonal variations in air density are noted at the altitude of about 70 km and

130 and 70% from the standard density of air at the latitude of 60°, and 110 and 95% -- at the latitude 30°. Seasonal variability at all latitudes decreases significantly above 70 km.

The annual trend of mean monthly values of air density at the altitudes 40, 50, and 60 km for 30 and 60° N. Lat is shown in Fig. 1.5 also in deviation from the 1952 US Standard Atmosphere /104/. The data in Fig. 1.5 show that the highest density of air at the latitude 30° is noted in May and June. The maximum shifts to June and July at the latitudes 60°. The lowest air density is noted at both latitudes in December and January. The amplitude of changes in the air density at latitudes 60° considerably exceeds the amplitude at the latitude 30°. Considerable deviations of the mean monthly density from the smoothed values denoted by the curves /24
occur at the latitude 60°, especially in autumn and winter. These deviations are evidently associated with abrupt warming in the stratosphere (see Section 1.4).

Differences between the maximum and minimum monthly air densities, which are observed -- as already noted -- in January and June, increase with latitude. Whereas this difference is 29% at the altitude 40 km and latitude 60°, it is 61% at the latitude 80°, that is, it increases by a factor of two. The trend in the seasonal variability of air density described in different latitudes of the northern hemisphere agrees well with the results obtained by V. G. Kidiyarova /27/.

Investigating the distribution of air density above 80 km involves major difficulties. They consist in the fact that there are only isolated cases in which meteorological rockets have risen to altitudes above 80 km, and various indirect methods, owing to their inadequate precision, afford only in approximate estimate of atmospheric density. One of the most exact indirect methods is the technique of determining density by means of observations of meteor trails.

Small solid particles -- meteors -- continuously fall into the earth atmosphere. Their number usually exceeds 150-200 per hour. On entering the atmosphere, the meteors heat up strongly as they decelerate in its relatively dense layers and are vaporized in the layer from 110 to 70 km. Here a growing trail (column) of strongly ionized air 40-50 km and several meters in diameter is formed, which can be photographed.

The principal equations of the physical theory of meteors make it possible to determine the density of the atmosphere in the meteor zone. If the deceleration of the meteor dV/dt is determined, the density of the atmosphere can be calculated by the formula /64, 65/:

$$\rho = -kM^{1/3}V^{-2}\frac{dV}{dt},$$

where k is a coefficient expressed by the formula

$$k = \Gamma^{-1}\pi^{-\frac{1}{3}}\left(\frac{4\delta}{3}\right)^{2/3},$$

and Γ is the coefficient of deceleration; δ is the density of the meteor particles; M is the mass of the meteor body; and V is the velocity of the meteor.

These quantities can be found from the brightness and other characteristics of meteors determined by photographs of meteor trails.

Fig. 1.6 shows the variation in air density with altitude obtained from 179 evaluations made on the basis of the above-mentioned technique /64, 65/ in Kiev. Here also presented is the variation in density with altitude based on measurements in Odessa (curve 1), as well as the values of air density based on measurements in Kiev when only data for meteors with velocities greater than 40 km/sec (curve 3) were considered. From Fig. 1.6 it follows that satisfactory agreement is observed in the altitude range 90 to 110 km. At lower altitudes the air density based on Kiev data is considerably higher than based on Odessa data. The overstatement of density values obtained from Kiev measurements is accounted for by the fact that here the air density was calculated from a relatively larger total of meteors whose velocity was lower than 40 km/sec, while the above-presented formulas are valid given the condition that the meteor velocity exceeds 40 km/sec /65/. If we do not include meteors with low velocity, the results of the determination of air density in these stations agrees much better. /25

Only isolated experiments have been conducted in studying diurnal variability of air density in the stratosphere and mesosphere. An example of these can be taken as the two launches of meteorological rockets made over Kwajalein Island with a 13-hour interval /114/.

The measurements show that the air density observed at noon time differs from the density observed at night. In the layer from 30 to 120 km, the atmospheric density in the day exceeded the night value of density by about 10%. Investigations with meteor trails showed that in the 80-110 km atmosphere the air density changes 20% from day to night in the middle latitudes.

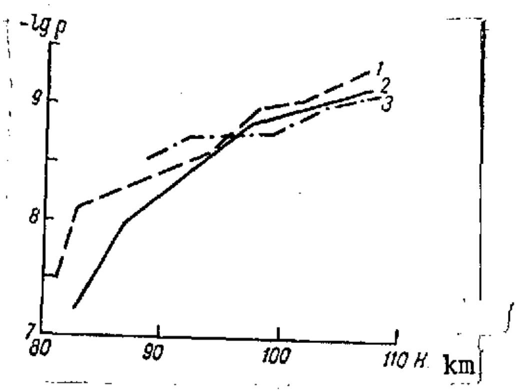


Fig. 1.6. The variation in air density with altitude determined from meteor brightness:
1. Odessa
2. Kiev (all meteors)
3. Kiev (meteors with V > 40 km/sec)

1.4. Abrupt Warming in the Stratosphere and the Associated Variations in Thermodynamic Parameters of the Atmosphere

In winter, the usual structure of the temperature field caused by radiative and advective-dynamic factors in the stratosphere and mesosphere is sometimes disturbed. This becomes evident above all in the rapid temperature rise at some level, which during a short time interval encompasses a large part of the stratosphere and rarely the lower half of the mesosphere. A sharp temperature rise in these atmospheric layers leads to quite large changes in pressure and air density. Warming up of the stratosphere was first detected by Sherhag over Europe in February 1952 [117, 118].

Some investigators state that warming in the stratosphere was first discovered in higher atmospheric layers. For example, during the warming period occurring in January-February 1958 [50], at the altitude of 40 km over Hays island, on 19 January the air temperature reached 290°K, while on 16 and 21 December 1957 it was 223 and 245°K, respectively. During this time the temperature at altitudes below 28-30 km differed little from the temperature usually observed here in winter. By 10 February 1958, the warming encompassed the atmospheric layer from 15 to 35 km in which an unusually high temperature was detected, 240°K. In similar fashion, warming developed in January 1960, January 1961, and so on [50]. /26

In radiosonde and especially rocket observations conducted in the past decade in the USSR and the United States, winter stratospheric warming was regularly detected. This phenomenon is observed nearly every year, and in some years even several times during the winter season.

Table 1.5 gives a list of warmings occurring in the high latitudes of the northern hemisphere at the altitudes 23-25 km in 1957-1964 according to Kh. P. Pogosyan and A. A. Pavlovskaya [24, 56, 57].

During the period of warming, air temperature often increases by a large value. At the altitude of 23-25 km, as shown in the table, the stratosphere becomes 20-40°C warmer, and during the January 1963 warming, there was a 56°C temperature rise. Observations showed, /27

TABLE 1.5. WARMINGS OVER THE ARCTIC AT THE ALTITUDES 23-25 KM (1957-1964)

Warming period	Duration, days	Temp., °C at beg of per	Temp., °C at end of per	Extent of warming, °C
3—9/XI 1957	7	—65	—25	+40
24/I—1/II 1958	8	—76	—41	+35
15—20/XI	5	—59	—47	+12
17—20/I 1959	3	—61	—41	+20
3—13/III	10	—61	—47	+14
2—14/I 1960	12	—60	—35	+25
5—11/II	6	—65	—49	+16
23/XI—3/XII	10	—69	—49	+20
19—27/XII	8	—65	—48	+17
3—13/I 1961	10	—79	—53	+26
26/II—8/III	10	—74	—37	+37
30/I—19/II 1962	20	—65	—34	+31
17—28/I 1963	11	—75	—19	+56
19/II—5/II 1964	13	—74	—32	+42

however, that maximum heating of the stratosphere usually occurs in the 30-40 km atmospheric layer. From the data of rocket sounding, the air temperature over Churchill station made in January 1958 increased by nearly 70°C at the altitude 40 km.

As the warming embraces increasingly lower levels, the upper stratospheric layers in which it began gradually cooled down.

Warming does not occur simultaneously in the entire upper-latitude stratosphere. Data from a rocket sounding in January 1958 showed that a temperature rise was noted at an altitude 40 km over Hays island on 19 January, while during this period ordinary winter temperatures were observed over Churchill station. Warming here began only on 25 January at the altitude of 45 km. Sherhag [11], in studying this warming, established that at the altitude of the 25 mb isobaric surface the thermal region shifted west by northwest. The thermal region at this level was discovered over Central Europe on 25 January. Later, it gradually shifted to the North Sea, on 30 January it reached Iceland, and then crossed North America.

Warmings in the stratosphere occur infrequently also in summer, however they are not as extensive as in winter. According to the data of Sherhag [11], abrupt warming began over Berlin on 7 July 1958 at the 20 and 25 mb isobaric surfaces; the abrupt warming reached its culmination on 10 July. During this period the air temperature rose 5°, then began to decrease, and reached its initial values. This pattern is characteristic for observations at a certain fixed point. In reality, the process of stratospheric warmings is highly complex and is associated with the radical arrangement of the thermobaric field at all altitudes.

Abrupt warmings in the stratosphere are not the exclusive feature of the polar latitudes. They penetrate the temperature latitudes all the way down to the 50-45° parallels. Cases in which warming was distributed even to 30° N. Lat have been recorded.

Significant transformations of the pressure field are accompanied by abrupt temperature rises in the period of winter warmings in the stratosphere. Cases have been recorded when as the result of abrupt temperature rises, the winter cyclonic type of pressure field in the stratosphere was replaced with a summer anticyclonic type.

Frequently a disruption in the stratospheric cyclone and its breakup can be seen, as well as a cold cell, into two separate cyclonic vortices, one of which is in the temperature latitudes, and the other in the subpolar. At the same time, the Pacific Ocean and Atlantic anticyclones become much stronger and are shifted northward. During the period of warmings in the middle stratosphere, the air temperature above several stations in Canada rose 65-70°C in five days, as a result of which the altitude of the 10 mb isobaric surface increased more than 800 m in the center of the Atlantic anticyclone. A maximum increase in the altitude of this isobaric surface was observed over Greenland and amounted to 2400 m /24/. At higher levels, there is any more sizeable rearrangement of the pressure fields. At the altitudes 40-55 km, anticyclones can merge into a single system lying where there had been a cyclonic vortex before the warming. The winter type of the pressure field is replaced by the summer /92/. /28

As a result of the significant rearrangement of the thermobaric field in the stratosphere during the period of winter abrupt warmings, there is a sharp increase in air density. An example of this can be taken as the warming in January 1958, when the air density over Greenland rose 13% at the altitude 29 km (Table 1.6).

TABLE 1.6. TEMPERATURE AND DENSITY OF AIR DURING A WARMING PERIOD IN THE STRATOSPHERE OVER CHURCHILL STATION IN JANUARY 1958 (AFTER I. P. VECHKANOV)

Meteorological element	Date	H, km								
		20	25	30	35	40	45	50	55	60
t °C	27/I	−55	−62	−71	−59	−33	−17	−13	−17	−23
	29/I	−55	−65	−65	−23	+10	+17	+ 9	− 4	−16
ρ g/m³	27/I	85.0	40.2	18.7	7.98	3.51	1.61	0.85	0.45	0.239
	29/I	84.0	40.2	17.6	7.30	3.51	1.87	1.10	0.65	0.363

From Table 1.6 it follows that in two days the air density increased 16% at the altitude 45 km, and 52% at the altitude 60 km. Cases were observed when during warming periods the air density at the altitude 50 km rose 70-80%, and sometimes 100%.

There are two hypotheses accounting for the cause of abrupt winter warmings in the stratosphere. The first of these is that of Sherhag [117, 118] and states that stratosphere warmings are a consequence of the manifestation of solar activity in the atmosphere. Sherhag correlated the trend in the density of the earth's magnetic field, which depends on solar activity, with temperature in the upper atmospheric layers and determined that a fairly close relationship is observed between them. In addition, it was found that during a period of intense warming in the atmosphere occurring in January 1958 there was an increase in the deceleration of the second artificial earth satellite, which was determined by the increase in air density at the satellite altitudes.

The presence of a correlation between solar activity and changes in temperature and pressure in the stratosphere was confirmed also by other investigators [10, 34, 40, 41, 66, 111, 116]. They maintained that the cause of the temperature rise during periods of abrupt warming is the heating of gases constituting the atmo- /29
sphere as a result of the absorption of particles of solar origin penetrating the stratosphere during solar flares.

The second hypothesis can be called the advective-dynamic hypothesis. Advocates of this hypothesis [56, 57, 80] dispute that direct solar manifestations exist in the mesosphere, stratosphere, and troposphere, asserting that the buildup of winter stratospheric warmings is affected jointly by vertical and horizontal air movements. In investigating the 1957 warmings, Craig and Lateev [91] calculated the vertical movements over the North American continent and the adjoining regions of the Atlantic. As shown by calculations, the maxima of the descending air motions were observed in the region of the Great Lakes at the 25 mb isobaric surface on 4 February (8 cm/sec), at the 50 mb isobaric surface on 6 February (6 cm/sec), and at the 100 mb isobaric surface on 8 February (4 cm/sec). The descending air current spread over vast areas. Even when there were descending motions of lower magnitude (3 cm/sec), according to the study by M. V. Shabel'nikova [80], with a vertical temperature gradient 0.4°/100 m observed on the average in the stratosphere, the adiabatic temperature rise was 25°C per day in the lower stratosphere and 36°C in the higher layers.

After a region of warm air forms in the stratosphere under the effect of the descending movements, advective transport takes on high significance. In the lower stratosphere by the end of the warming period this can amount to 13-17°C per day [56].

The above presented results account mainly for the buildup of atmospheric abrupt warmings and not the cause of this phenomenon.

At the present time the concept /49/ that there is a relationship between stratospheric warmings and large-scale macroturbulent formations which arise in the troposphere and then encompass the stratosphere by penetrating increasingly higher layers is continuing to be developed.

CHAPTER TWO /30

STRUCTURE OF WIND FIELD IN THE TROPOSPHERE, STRATOSPHERE, AND MESOSPHERE

2.1. Wind Regime in the Troposphere, Stratosphere, and Lower Mesosphere

Characteristics of wind field in the troposphere, stratosphere, and mesosphere can be judged from the structure of the pressure field, which indicates that the westerly transport of air masses predominates in the troposphere of the northern and southern hemispheres in the middle and high latitudes in winter and summer; this air-mass transport is perturbed in places by acquiring a northerly or southerly component. This becomes manifested in maps of baric topography in the form of troughs and ridges.

The velocities of the westerly flow in winter and summer differ, especially in the northern hemisphere. In a map of the topography of the 500 mb surface, the horizontal gradients of the geopotential have a considerably smaller value in summer than in winter. In the transitional seasons intensive westerly air flows also predominate in the troposphere.

The structure of the wind field in the troposphere is inhomogeneous. This inhomogeneity is manifested in that the wind velocity as a rule rises with altitude in this atmospheric layer. The maximum wind velocities are usually observed close to the tropopause. In addition, the wind velocity changes with latitude, rising or falling with increase in the latitude. Thus, close to the tropopause there arise zones in which the wind velocities reach considerable values. These zones are relatively narrow in width and relatively long in length and are called jet streams.

Jet streams in the troposphere are caused by the large contrasts in temperature and lying in the zone of transition from the high cold cyclones to the warm high anticyclones. From the energy point of view, jet streams are zones of maximum reserves of kinetic energy. Wind velocities in jet streams usually exceed 30 m/sec, sometimes reaching 70-100 m/sec and higher.

Study has shown /11/ that in the northern hemisphere four planetary high frontal zones are observed: the arctic, the northern of temperate latitudes, the southern of temperate latitudes, and the subtropical. In winter the first of these is located on the average around the latitude 68° and has a mean wind velocity along the zone axis of 23.4 m/sec. The northern and southern planetary high-altitude frontal zones are noted at the latitudes 56 and 39°, with mean wind velocities along the axes of 38.6 and 55.8 m/sec, respectively. The subtropical frontal zone is situated at the latitude 29° with a mean wind velocity along the axis of 64 m/sec. /31

In summer, the mean latitude of all four planetary high-altitude frontal zones increases. The high-altitude arctic frontal zone lies at the latitude 73°, the northern of temperate latitudes -- at the latitude 64°, the southern of temperate latitudes -- at the latitude 48°, and the subtropical -- at the latitude 41°. Mean velocities along the axes of these zones are 18.1, 27.9, 37.1, and 44.4 m/sec, respectively.

Relatively weak and unstable streams are observed in the troposphere in the equatorial and tropical zones. In the lowest layer, as a rule, easterly currents predominate, changing at the altitudes 5-10 km to westerly currents, which in the upper troposphere often again changing into easterly.

In the stratosphere and mesosphere, the nature of air streams is also determined by the structure of the pressure field. In winter, much of the northern hemisphere is covered by a cyclonic vortex, whose center lies near the pole (see Fig. 1.4). In summer, in contrast, anticyclonic circulation is observed over a great portion of the hemisphere. The center of the high pressure region also lies over the polar regions (see Fig. 1.3). Thus, in winter the westerly wind observed in the troposphere continues to predominate in the stratosphere and mesosphere. In summer, the westerly wind at the altitude of about 20 km changes into an easterly wind. The latter propagates up to the upper limit of the mesosphere.

Characteristics of the annual trend of the zonal component of wind velocity in the stratosphere, mesosphere, and lower atmosphere well illustrate the time-based vertical profiles (Figs. 2.1 and 2.2) plotted from the data of atmospheric rocket sounding for 30 and 60° N. Lat. /103/. In these profiles, isolines above 80 km, being unreliable, are drawn with a dashed line.

Both at 30° and at 60° N. Lat the change of the winter westerly current into the summer easterly current as a rule occurs from the top down. It begins in the upper mesosphere at the end of March to the beginning of April and continues through April in the lower mesosphere and stratosphere.

A comparison of these results shows that the circulation of the wind spreads downward in the zone of 60° N. Lat faster than in the zone of 30° N. Lat. In addition, stronger winds are found in the zone of 30° N. Lat than in the zone of 60° N. Lat. The autumn restructuring of the wind field occurs in an especially short time period. Above 80 km circulation and strengthening of wind occurs again in all seasons. Particularly strong winds in the lower thermosphere are observed in summer.

/32

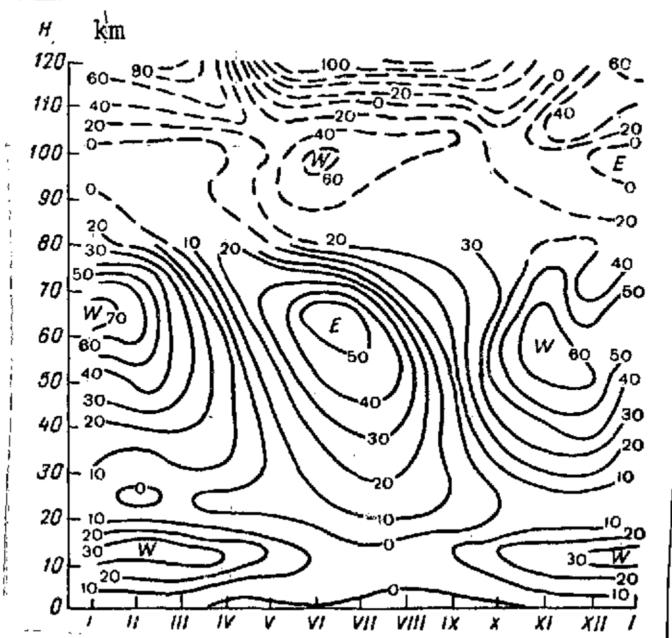


Fig. 2.1. Mean monthly zonal components of wind velocity (m/sec) at 30° N. Lat

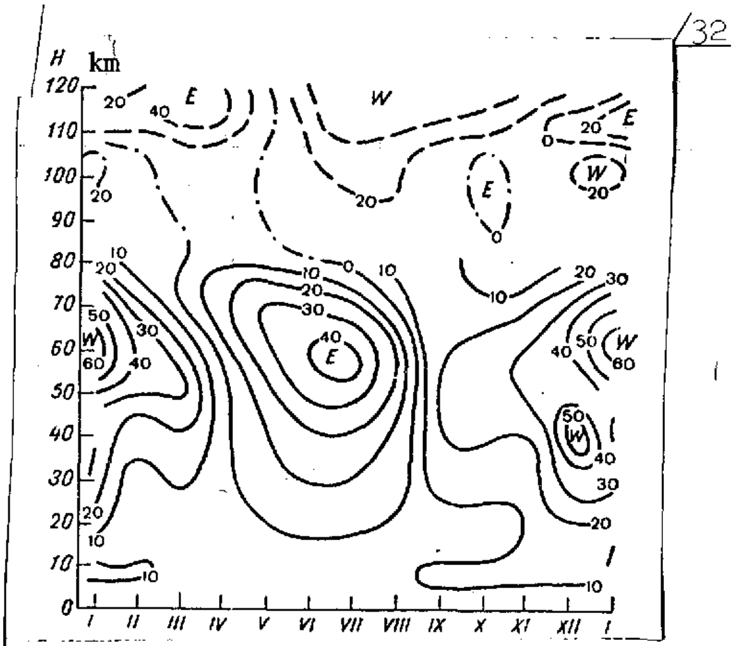


Fig. 2.2. Mean monthly zonal components of wind velocity (m/sec) at 60° N. Lat

The vertical time-based profiles of zonal components of wind velocity show the features of periodic oscillations in the velocity caused by seasonal changes in the pressure fields. Still, during summer there are frequent perturbations of the winter regime of circulation associated with abrupt warmings in the stratosphere. For example, on 9 February 1962 during a warming period in the 25-50 km layer, strong westerly winds typical of the winter regime were replaced with easterly winds. Over Churchill station on 19 January and 19 February 1962 a westerly wind /33 at a velocity of 47 and 30 m/sec, respectively, was observed at the altitude 37 km /24/. At the end of January and the beginning of February, an easterly wind with a velocity of about 40 m/sec became evident.

In some cases the disturbances in the usual winter regime of circulation are manifested not in the circulation of a westerly wind, but in its abrupt, short-lived weakening, or in weakening and replacement by the easterly direction only in a small atmospheric layer.

Investigations showed that the stability of the zonal and meridional components decreases with altitude.

Harmonic analysis of three series of atmospheric soundings made with meteorological rockets over the Eglin (Air Force Base) in May 1961 and White Sands stations in February 1964 (about 30° N Lat) showed the presence of diurnal and semidiurnal variations in wind components that were considerable in magnitude. Table 2.1 presents their amplitudes in the 30-60 km atmospheric layer /110/.

TABLE 2.1. AMPLITUDES OF DIURNAL AND SEMIDIURNAL VARIATIONS OF WIND VELOCITY (M/SEC)

H km	Zonal component				Meridional component			
	Eglin AFB, May 1961		White Sands Feb 1964		Eglin AFB, May 1961		White Sands Feb 1964	
	diurnal	semidiurnal	diurnal	semidiurnal	diurnal	semidiurnal	diurnal	semidiurnal
30	2.1	0.9	0.2	0.8	1.4	0.8	0.3	1.7
35	0.6	0.8	4.7	2.5	1.9	1.9	2.5	0.8
40	1.9	2.5	5.9	2.3	1.3	1.9	2.0	0.9
45	6.7	1.4	13.2	5.1	7.8	2.4	10.0	2.1
50	3.4	1.9	3.1	1.6	7.7	2.0	1.9	3.3
55	1.5	2.8	6.4	3.1	4.5	3.1	5.5	1.7
60	2.4	2.0	12.1	3.9	5.4	2.2	1.3	6.3

For comparison, Table 2.2 gives the mean-annual diurnal and semidiurnal variations of the meridional and zonal components of wind velocity in the troposphere and lower stratosphere obtained from radiosonde observations /99/.

From Tables 2.1 and 2.2 it follows that diurnal and semidiurnal variations in wind velocity components are small in the upper troposphere and in the lower stratosphere to 30 km; this indicates the relatively low variability of wind with time. A rise in the amplitude of diurnal and semidiurnal variations accompanies a further

TABLE 2.2. MEAN-ANNUAL AMPLITUDES OF DIURNAL AND SEMIDIURNAL VARIATIONS IN ZONAL AND MERIDIONAL COMPONENTS OF WIND VELOCITY. LAJES FIELDS, AZORES (m/sec) /34

H km	Zonal component		Meridional component	
	diurnal	semidiur.	diurnal	semidiurnal
9,2	0,2	0,5	0,1	0,1
11,8	0,6	0,5	0,2	0,2
16,1	0,3	0,6	0,3	0,3
20,6	0,2	0,4	0,3	0,4
23,9	0,2	0,7	0,2	0,6
28,5	0,3	0,9	0,7	0,6

increase in altitude, where the diurnal amplitudes as a rule exceed the semidiurnal. Amplitudes of variations in wind velocity components have a maximum at the altitude 45 km, and in winter it is greater than in summer; and the minimum is at the altitudes 50-55 km. Higher up, a trend to increased diurnal and semidiurnal amplitudes is observed. Above 60 km, as will be shown below, wind variability rises considerably.

2.2. Distribution of Wind in the Layer of Meteor Trails

Meteorological rockets carry out atmospheric sounding usually at altitudes to about 60-70 km. In the higher atmospheric layers measurements of meteorological elements, including wind, occur only episodically with rockets, and at altitudes exceeding 70 km they are nearly entirely absent. Accordingly, when investigating the wind regime in the 80-110 km atmospheric layer, which is customarily called the meteoric layer, wind measurement by means of meteor trails plays an important role [3, 23, 25, 97]. This method has adequate precision not only for determining the rate of transport, but also for investigating the turbulent structure of wind.

Measurement of wind by means of meteor trails is based on the following principle. As already mentioned, when entering the dense atmospheric layer the meteor heats up strongly and vaporizes, forming a meteor trail -- a column of highly ionized air. If the meteor trail formed is irradiated with high-frequency radio waves, a point at which the wave encounters the trail at a right angle reflects the signal. Under the action of wind, the meteor trail /35 is shifted and the total distance from the observation site to the point producing the reflection changes. This leads to a Doppler frequency shift, which yields a change in the phases of the reflected signals proportional to the radial component of wind velocity. The altitude at which the radio echo appears and, thus, the wind is measured depends on the wavelength sent by the radar. For a wavelength of 9 m, for example, it corresponds to 93-95 km.

The meteor layer is characterized by very intense turbulent mixing. Therefore to analyze the wind regime one must calculate the mean-hourly wind velocity components, permitting the diurnal trend of wind velocity to be investigated. Observations showed that the greatest scatter of wind velocity components occurs in those hours when there is a change in wind direction. Table 2.3 gives the meridional components of wind velocity from observations in Khar'kov from 18:00 to 24:00 local time /25/. Positive values as before correspond to the wind direction from south to north.

TABLE 2.3. MERIDIONAL COMPONENTS OF WIND VELOCITY (m/sec). KHAR'KOV, 1964

Date	Time, hours					
	18–19	19–20	20–21	21–22	22–23	23–24
9/VI	25	5	–12	–28	–40	–45
11/VI	35	–14	–13	–30	–29	–44
13/VI	10	24	13	–27	–33	–43

Harmonic analysis of the mean-hourly wind velocities reveals the prevalent wind, and also diurnal and semidiurnal variations. Of greatest interest are the values of the prevalent wind, as well as semidiurnal variations, since the amplitudes of the latter considerably exceed the amplitudes of the other harmonics. Investigations showed that for Moscow, for example, the amplitude of semidiurnal variations of the zonal component have their greatest value in the winter months (32 m/sec) and decrease by a factor of 1.5-2 by June /3, 25/. Amplitudes of semidiurnal variations of meridional components in January-February are approximately the same. In March and May their values decrease to 10-18 m/sec, while in April and June they approach the winter values for the zonal components.

The zonal components of the prevalent wind reach their maximum of 20-30 m/sec in April and June. In January and February they are 1-5 m/sec, and in May -- 12-15 m/sec. The sign of the zonal components also changes during the year. An easterly stream is observed in February and March, in April-May, it changes to the westerly and again becomes easterly in June. /36

Fig. 2.3 gives the average results of wind measurements /25/. From Fig. 2.3 a it follows that variations in the amplitudes of the semidiurnal harmonics of wind velocity components shows similar trends in all of the stations listed, that is, their values are observed to decrease toward spring and summer. From Fig. 2.3 b it is clear that the behavior of the zonal components (v_z) in all three stations is approximately the same. Similarity can also be noted in the nature of the changes of meridional components (v_m) as well. Therefore, air currents in the meteor zone over Manchester, Moscow, and Khar'kov have much in common. This is actually not surprising since the difference between the latitudes of these stations is small.

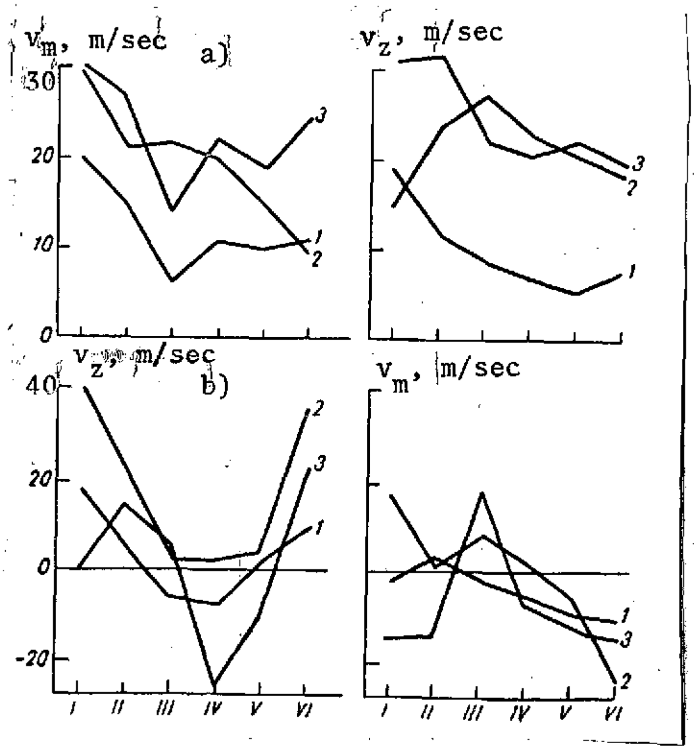


Fig. 2.3. Variations in the Values of Semidiurnal Harmonics (a) and Prevalent Winds (b) in the Meteor Zone:
1. Manchester, 1953-1958
2. Khar'kov, 1960-1961
3. Moscow

These data indicate that considerable wind velocities are observed in the 80-110 km atmospheric layer. Also, in this layer wind exhibits greater variability in magnitude and direction, indicating highly developed turbulence. /37

2.3. Turbulent Motions in Dense Atmospheric Layers

Different air mass transport rates are observed in the atmosphere at different levels. In addition, air mass transport is thermally inhomogeneous both in the vertical and in the horizontal directions. All this leads to conditions favorable for the buildup of turbulence.

There are two approaches to the study of turbulent motions. The first is the semiempirical method of investigation. In a turbulent flow regime the velocity vector at some point in space changes with time. Therefore when studying turbulent motions it is convenient to subdivide the field of instantaneous velocities into mean velocities ($\bar{u}$, $\bar{v}$, $\bar{w}$) and velocity pulsations (u', v', w'), averaging the equations of motion in accordance with known principles of hydrodynamics. As a result of averaging, terms appear in the equations of motion containing quantities consisting of the fluctuational velocities

$$-\rho\overline{(u')^2},\ -\rho\overline{u'v'},\ -\rho\overline{w'u'};$$
$$-\rho\overline{(v')^2},\ -\rho\overline{v'w'},\ -\rho\overline{(w')^2}.$$

These quantities express the transport of momentum of individual air masses owing to pulsational motions and are called turbulent stresses. If one considers that there is an analogy between turbulent and molecular motions, then by using some semiempirical relations one can obtain the quantities of turbulent stresses. For example,

$$-\rho\overline{u'w'} = \rho k_z \frac{\partial \bar{u}}{\partial z}.$$

Thus, turbulent stresses can be expressed in terms of characteristics of mean motion. The quantity k_z characterizes the intensity of momentum transport in the vertical direction as a result of fluctuational motions and is called the coefficient of turbulence.

The semiempirical theory of turbulence came to be applied in solving a number of problems of atmospheric physics, aerodynamics, and other sciences. But it cannot be used in investigating phenomena in which internal structural properties of a flow are determining. In this case another approach -- the statistical -- is used.

Turbulent pulsations are random variables. Therefore to determine structural features of turbulent flow it is required to bring in statistical methods of investigation and, therefore, to describe the structure of turbulent motions the apparatus of the theory of random functions must be used. A. A. Fridman and L. V. Keller used for this purpose the covariance functions /38

$$R_{jk}(x_1, x_2, x_3, t, \xi_1, \xi_2, \xi_3, \tau) = M\left[u'_j\left(x_1 - \frac{\xi_1}{2};\ x_2 - \frac{\xi_2}{2};\ x_3 - \frac{\xi_3}{2};\ t - \frac{\tau}{2}\right) u'_k\left(x_1 + \frac{\xi_1}{2};\ x_2 + \frac{\xi_2}{2};\ x_3 + \frac{\xi_3}{2};\ t + \frac{\tau}{2}\right)\right]. \quad (2.1)$$

The dependence of the function on the components $\xi_1, \xi_2, \xi_3, \tau$ expresses the internal structural properties of the flow, while its dependence on the variables x_1, x_2, x_3, and t characterizes the difference between the external conditions for different parts of the medium. A turbulent medium is anisotropic. This means that in the general case the directions x_1, x_2, and x_3 are not equivalent in a turbulent flow. However, within a flow a volume can be singled out which all directions of the coordinate axes are equivalent. This property of the turbulent flow is called local isotropy. If it is understood, in addition, that the external conditions under which the motion of the medium occurs are the same, then the statistical characteristics of the medium will be identical for all its points and, therefore, the turbulent flow can be regarded as homogeneous, and the velocity of the medium can be regarded as a stationary random function. The covariance function for this flow depends only on the difference in the arguments $\tau = t_2 - t_1$:

$$R_u(\tau) = M[u'(t)\,u'(t+\tau)], \tag{2.2}$$

if the motion at one point is considered, but in different time intervals, and

$$R_u(l) = M[u'(x)\,u'(x+l)], \tag{2.3}$$

if the motions considered at different points of the flow separated from each other by the distance $l = l_2 - l_1$.

The variables τ and l are correspondingly the time and linear scales of turbulent motions. A hypothesis states that turbulent eddies are transported by the mean flow. In addition, it can be assumed that the time and linear scales are related in terms of the velocity of the mean motion $l=\bar{u}\tau$.

In a study of the properties of the turbulent flow, very often use is made of the spectral densities of turbulent pulsations, which constitute a Fourier transform of the covariance function

$$S_u(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_u(\tau)\,e^{-i\omega\tau}\,d\tau, \tag{2.4}$$

where ω is frequency.

The dispersion of the turbulent wind pulsations is associated with the spectral density. This relationship has the following form: /39

$$D_u = R_u(0) = \int_{-\infty}^{\infty} S_u(\omega)\,d\omega. \tag{2.5}$$

In several cases the structure function

$$B_u^2(\tau) = M\{[u'(t) - u'(t-\tau)]^2\}. \tag{2.6}$$

is very convenient in characterizing the internal properties of the turbulent flow.

The relationship between structure and covariance functions is expressed by the obvious equation

$$B_u^2(\tau) = 2R_u(0) - 2R_u(\tau), \tag{2.7}$$

and the relationship between structure and correlation functions is expressed by the relation

$$B_u^2(\tau) = 2R_u(0)\,[1 - r_u(\tau)]. \tag{2.8}$$

The statistical theory of turbulence was elaborated in works by A. N. Kolmogorov, A. M. Obukhov, M. I. Yudin, J. B. Chaylor, et al. A. N. Kolmogorov showed /28, 29/ that the main energy source of fluctuational motion is the instability of the mean motion. The laminar motion of a viscous liquid with characteristic v and characteristic scale L is steady-state, when the Reynolds number $Re = \frac{vL}{\nu}$, where ν is the kinematic viscosity of the liquid, does not exceed the critical value Re_{cr}. If $Re > Re_{cr}$, the motion becomes unstable. Velocity fluctuations v_l' arise in the flow, whose linear scale is l. The specific energy of these fluctuations is $(v_l')^2$. Therefore, in the inception of fluctuations in velocity from the mean value, energy proportional to $\frac{(v')^3}{l}$ is transmitted per unit time. On the other hand, some of the energy of the fluctuational motion is dissipated. This part of the energy is proportional to $\frac{\nu (v')^2}{l^2}$. Then the condition for the existence of velocity fluctuations can be described as follows:

$$\frac{(v_l')^3}{l} > \nu \frac{(v_l')^2}{l^2}$$

or after certain transformations

$$Re_l = \frac{l v_l'}{\nu} > 1. \tag{2.9}$$

Since these calculations were made to a precision of specific numerical cofactors, it is proper that the solution can be written as: /40

$$Re_l > Re_{cr}. \tag{2.10}$$

The number Re_l is called the internal Reynolds number. From Eq. (2.9) it follows that large eddies for which a large Re_l is characteristic arise most easily. When inequality (2.10) is satisfied, these large eddies become unstable and pass on their energy to smaller eddies, which subsequently also lose stability. Thus, the transfer energy from the mean motion to fluctuational motion consists in the breakup of large eddies into finer ones. Here the dissipation of energy ϵ is large only for small eddies. If we use dimensional analysis, we can determine the scale of eddies λ whose fluctuational energy changes into thermal. This scale depends on the kinematic viscosity and the dissipation of energy

$$\lambda = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}. \tag{2.11}$$

In local-isotropic turbulence, as shown by A. N. Kolmogorov /29/, the following equality is valid:

$$B_u^u(l) = C\bar{\varepsilon}^{2/3} l^{2/3}, \tag{2.12}$$

in which C is the coefficient of proportionality. This relationship came to be called the law of "two-thirds".

The law of "two-thirds" was first derived by A. M. Obukhov /47, 48/ by the spectral decomposition of the velocity of a steady-state turbulent flow. To the law of "two-thirds" there corresponds a power dependence of the spectral density of turbulent pulsations on frequency, of the form

$$S(\omega) = C_1 \bar{\varepsilon}^{2/3} \omega^{-5/3},$$

which sometimes is called the law of "minus five-thirds."

M. I. Yudin /84/ investigated the applicability of the law of "two-thirds" and determined the effect of anisotropy of turbulent motions on the correlations of the wind field structure.

If we examine a large-scale flow, even at a distance of about 1 km the moduli of the differences in the wind velocity components along the horizontal proved to be sometimes less than along the vertical. However, if we limit ourselves to considering these differences only in the horizontal direction, we need not take the vertical inhomogeneity into account. M. I. Yudin obtained the following structure law for large-scale anisotropic turbulent motions:

$$E'_\phi(p_1) = \left(\frac{\lambda_1 E''_\phi \beta}{2-\beta}\right)^{-1/2} C^{-1} \bar{\varepsilon} p_1^{-1}, \tag{2.13}$$

where $E'_\phi(p_1)$ is the fluctuational energy of horizontal motion; /41
$E''_\phi(p_1)$ is the fluctuational energy of vertical motion, assumed constant, if the neighboring atmospheric layers are separated from each other by distances that are commensurable with the "mixing path"; λ_1, β, and C are constants; $\bar{p}_1$ is a quantity that

is the reciprocal of the scale of turbulent eddies. This structure law came to be called the law of "first power."

Numerous investigations dealt with an experimental check on the structure laws of turbulent motions, confirming the correlation that had been obtained theoretically. Several methods are used in the experimental investigation of turbulence in the free atmosphere. The greatest use was made of the method based on employing as the wind gust gauge an accelerograph mounted on an aircraft. Using accelerographs recording g-loads that arise when an aircraft flies in a turbulent flow, it is possible to determine the velocities of vertical wind gust.

The paper [85] presents a series of structure functions $B^2(\tau)$ of vertical wind gusts. The scale of the eddies at which the structure function reaches saturation is called the characteristic scale. Calculations showed that in 70% of the cases the B^2_{max} values occur in time interval τ_{char} equal to 3.5-7.5 sec, which corresponds to a linear eddy scale of the 0.6-2.0 km. The structure function of wind gust is conveniently approximated with the expression

$$B^2(l) = Al^n. \qquad (2.14)$$

The results of calculating the exponent n showed that its value on the average follows in the range 0.6-0.8. Thus, the feasibility of the law of "two-thirds" is confirmed for eddy scales of the order of 10 km.

Investigations showed [85] that the characteristic scale of turbulent eddies depends on the thermal state of the atmosphere, which can be characterized by the difference of the adiabatic γ_a and actual γ vertical temperature gradients. Up to an altitude 12 km, this function is determined by the expression

$$\tau_{char} = 1.5(\gamma_a - \gamma)^{-1.2}.$$

The structure of horizontal turbulent pulsations in the 6-12 km atmospheric layer is investigated by means of a Doppler system [52, 53]. The degree of general perturbability of the wind velocity field was estimated by means of the relative (with respect to the mean wind velocity at the given altitude) root mean square deviation σ_u

$$\psi = \frac{\sigma_u}{u}.$$

As shown by the calculations, at altitudes 6-12 km and wind velocities exceeding 50-60 km/hr, ψ fluctuates in the range 0.05-0.30. It has two maxima, one of which lies in the 7.5-8.5 km layer, and the other -- in the 9.5-10.5 km layer. The structure functions of horizontal turbulent pulsations also increase /44 with increase in the eddy scale, and their characteristic scale is approximately one order greater than the characteristic scale of vertical pulsations, averaging 18 km. If we approximate these structure functions by Eq. (2.14), at all altitudes within the 6-12 km layer the exponent n on the average will be close to 2/3, that is, the horizontal turbulent pulsations in the troposphere also obey the Kolmogorov-Obukhov structure law.

However, as shown by investigations, the exponent n and the coefficient A of the structure law of turbulent pulsations depend on the degree of thermal stability of the atmosphere. The exponent n drops with increase in γ. The rate of its decrease is large in the region where $n \geqslant 0.8$, and smaller $n < 0.8$. The coefficient A, in contrast, increases with increase in γ. This derives from the fact that as the instability is increased, the rate of dissipation of turbulent energy rises.

/42

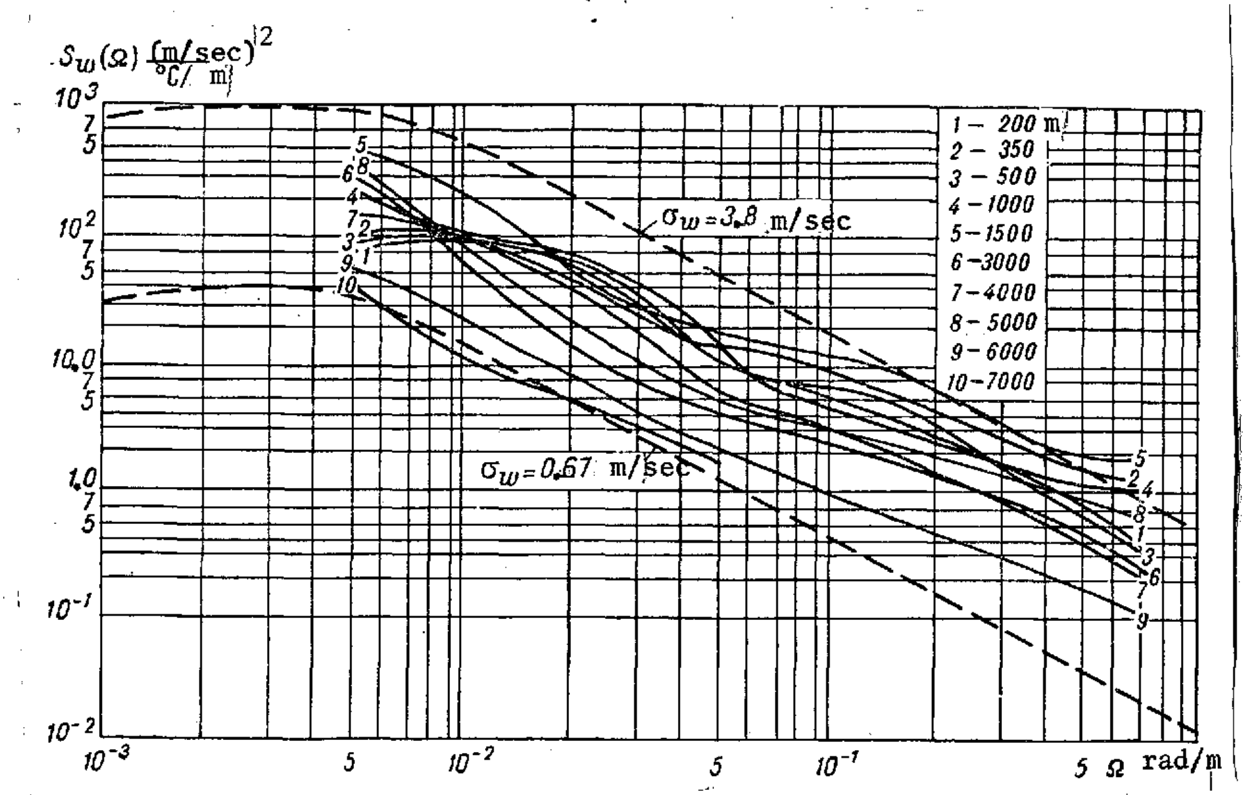


Fig. 2.4. Spectral densities of vertical wind gusts

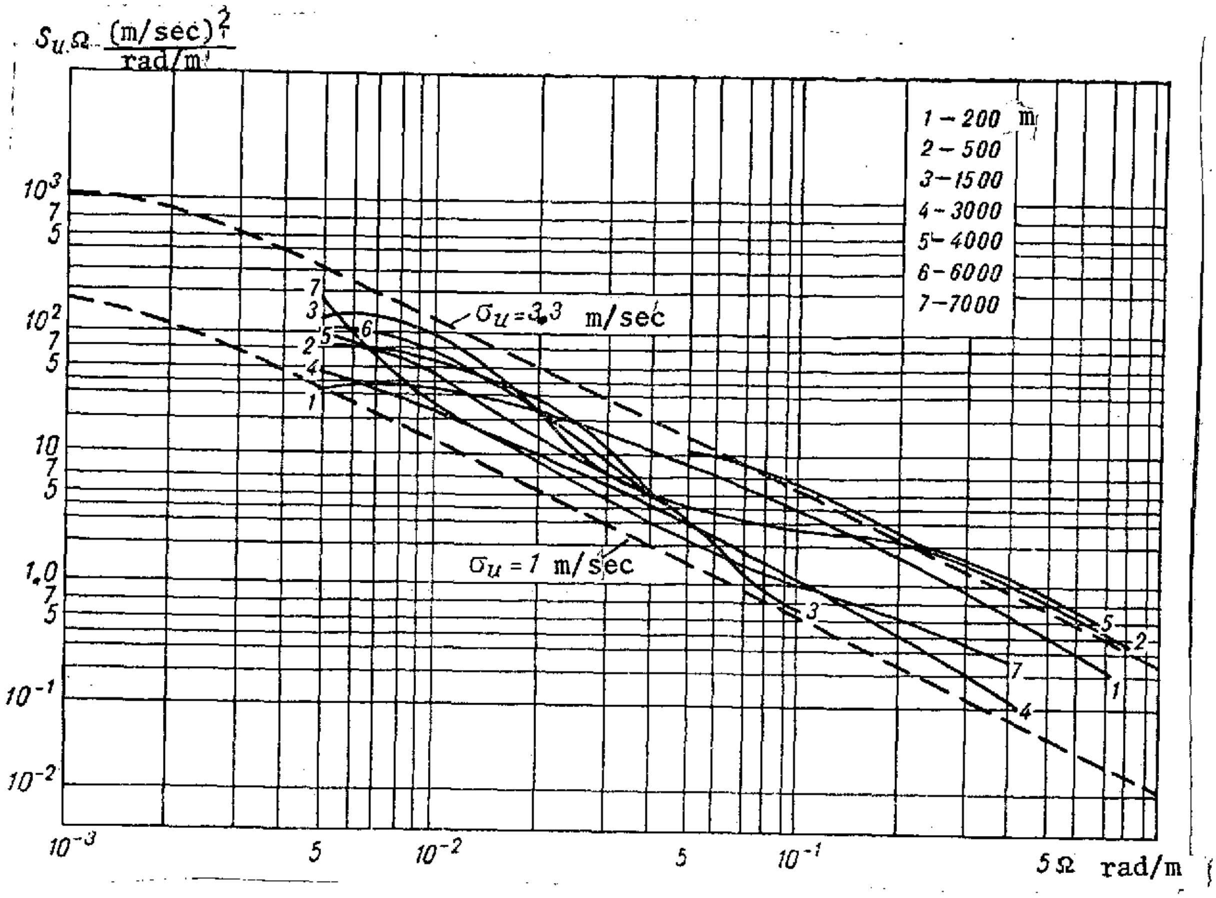


Fig. 2.5. Spectral densities of horizontal wind gusts
KEY: A. (m/sec)^2/(rad/m)
B. m/sec
C. rad/m

At the present time much attention is being given to investigating correlation and spectral characteristics of turbulent motions. Spectral densities of vertical $S_w(\Omega)$ and horizontal $S_u(\Omega)$ turbulent pulsations within this atmospheric layer, 0.2-7 km, obtained by G. P. Il'in, shown in Figs. 2.4 and 2.5. The spectral density of the vertical gusts varies over a broader interval than the horizontal gusts. If the spectral densities are integrated over all frequencies, we can obtain minimum and maximum dispersions of the turbulent components of wind velocity. Figs. 2.4 and 2.5 present the root mean square deviations corresponding to these dispersions.

A detailed study of the energy spectrum of turbulence was made in jet streams /82, 83/. Fig. 2.6 presents in logarithmic coordinates the spectral density in a jet stream and the altitude 8 km. The curve in this figure can be approximated with two linear segments. In different spectral regions, the spectral density is described by different power expressions of the form $S(\Omega) \sim \Omega^n$. For frequencies corresponding to scales less than 600 m, the exponent n = -1.67. This shows that the energy spectrum for these scales agrees well with the law "minus five-thirds." For larger scales, the exponent n = -2.7. The reason for the deviation of the spectral density from this law in this interval of scale values is the work that must be done by the turbulent eddies against the Archimedean forces, as a result of which the kinetic energy of turbulence for the case of stable temperature stratification the atmosphere changes into potential energy [82]. For small-scale eddies, the energy loss in work done against the Archimedean forces is negligibly small, while large-scale eddies /45 can lose a considerable fraction of the energy. Here the energy equilibrium within the inertia interval is disrupted, which is the reason for an increase in the slope of the curve in the region of large scales.

Measurements of turbulent motions were organized in the Central Aerological Observatory using radiosondes with an accelerometer accessories especially designed for this purpose. The observations were taken in Moscow, Sukhumi, and Tashkent. The resulting data permit several characteristics of wind gusts to be determined.

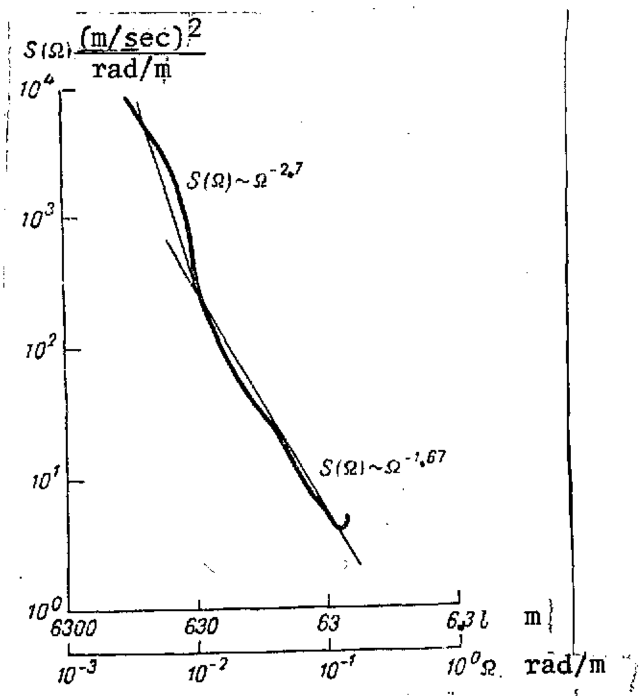


Fig. 2.6. Energy spectrum of turbulence occurring when jet streams intersect (7 February 1962)

Fig. 2.7 shows the incidence of turbulent pulsations with altitude. The incidence was calculated for each kilometer atmospheric layer, and the latter was assumed to be turbulent if a layer of turbulence 50 m thick was encountered in it. Incidence values obtained with the aid of radiosondes were compared with incidence values determined by aircraft sounding of the atmosphere (curves 2 and 3 in Fig. 2.7 a). It turned out that the incidence of turbulence calculated from aircraft measurements is smaller than the incidence obtained with radiosondes (curve 1 in Fig. 2.7 a). Averaged over the year and in all seasons, the incidence of turbulent pulsations falls off with altitude, reaching a minimum at the

altitudes 7.5 km in winter (curve 1 in Fig. 2.7 c) and 10-12 km in the remaining seasons. Then an abrupt rise in incidence occurs. /46
Averaged over the year (Fig. 2.7 a), autumn (Fig. 2.7 b) and winter (Fig. 2.7 c) its maximum is noted at an altitude of about 12.5 km and reaches 50, 50, and 70%, respectively. A further increase in altitude is associated with the decrease in the incidence of turbulence. An exception is represented by spring (Fig. 2.7 d) when a zone of high incidence of about 60% is observed in the 15-25 km layer. Turbulent motions weaken at an altitude of about 30 km. However, in winter the second maximum lies in this altitude. Studies showed that over Moscow, for example, in 30-40% of the cases moderate turbulence is encountered in the troposphere in summer, but in winter and spring -- in the stratosphere in the 15-20 km layer. Also in this layer is intense turbulence observed in 8% of the cases.

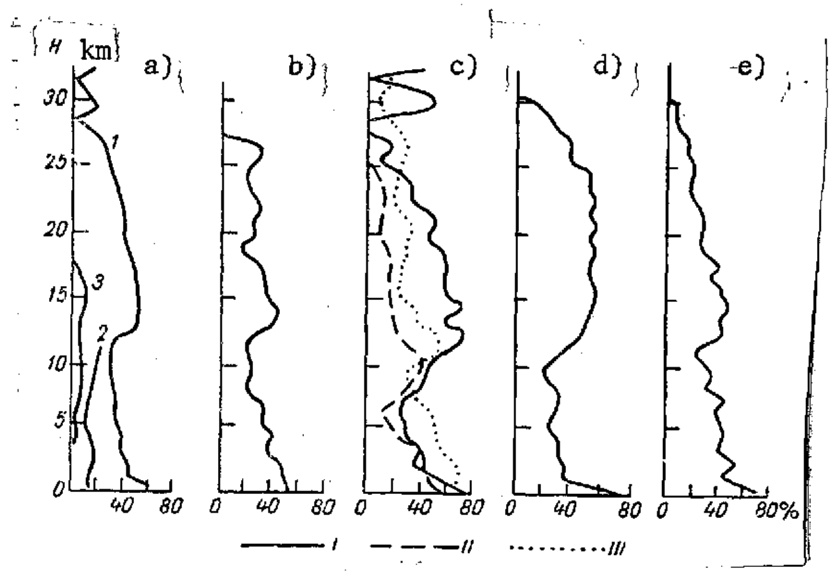


Fig. 2.7. Altitude distribution of incidence of turbulence over the year (a), autumn (b), winter (c), spring (d), and summer (e):
1. Moscow
2. Tashkent
3. Sukhumi

Radiosonde measurements made it possible to determine the thickness of the turbulence layers. The greatest thickness of layers with turbulence occurred over Tashkent, and the thinnest -- over Moscow. Aircraft studies confirmed the dependence of the incidence of large thicknesses of layers with turbulence on latitude. Whereas in the upper and middle latitudes the incidence of turbulent layers more than 1000 m thick was 10-15%, in the southern latitudes it rose to 30%. In the high and middle latitudes the maximum incidence occurred for turbulent layer thicknesses of 300-600 m, and the lowest -- for thicknesses of 400-800 m.

A characteristic feature of turbulence is its patchy character. This is especially true of the upper troposphere and the lower stratosphere. Zones of turbulence are extended horizontally up to 100-150 km. /47

The characteristics of turbulence examined below apply to the troposphere and the lower stratosphere. The development and improvement of atmospheric rocket sounding permitted experimental study of turbulent eddies throughout the atmospheric layer. To obtain characteristics of turbulent motions in this case, a sliding mean of the horizontal components of wind velocity were calculated by averaging wind profiles for some interval of altitude H, using the equality

$$\overline{v_i(z, t, h, H)} = \frac{1}{H} \int\limits_{z-\frac{H}{2}}^{z+\frac{H}{2}} v_i(z, t, h)\, dz, \tag{2.15}$$

in which $v_i(z, h, t)$ is the i-th velocity component (mean) of the wind, measured at the vertical interval h. Therefore, turbulent components of wind velocity can be calculated with the equality

$$v_i'(z, t, h, H) = v_i(z, t, h) - \overline{v_i(z, t, h, H)}. \tag{2.16}$$

Equality (2.16) makes it possible to obtain a set of pulsations of wind velocity components for each wind profile, on the basis of which the correlation function

$$r_i(\xi, t) = \frac{M\left[v_i'(z, t)\, v'(z+\xi, t)\right]}{M\left[v_i'^2(z, t)\right]}, \tag{2.17}$$

can be obtained, where ξ is the altitude interval, having the significance of turbulence eddy scale.

The set of correlation functions obtained for individual wind velocities can be averaged. Thus, it is possible to obtain a time-averaged correlation function of the turbulent wind velocity components, for example, for each season

$$r(\xi) = \frac{1}{N} \sum_{j=1}^{N} r_j(\xi, t). \tag{2.18}$$

If we apply Fourier transform (2.4) to equality (2.18), we get the energy spectral pulsations of wind velocity components.

The energy spectra of turbulent components of zonal and meridional wind velocity components in the atmosphere to an altitude of

50 km, based on the above-presented method, were calculated by Kao and Sands /101/ by processing 210 wind profiles obtained with the atmospheric rocket sounding conducted over White Sands station in the period from January 1963 to December 1964. These profiles include 21,000 wind measurements. The energy spectra are shown in Fig. 2.8, from which it follows that normalized spectral densities of pulsations in zonal and meridional wind velocity components are similar and proportional to Ω^{-2}. The maximum turbulent energy occurs at frequencies in the range 0.03-0.06 km^{-1} (eddy scale 2.5-20 km). /48

Table 2.4 gives the dispersions of turbulent components of zonal v and meridional u components of wind velocity for seasons and averaged over the year.

TABLE 2.4. SEASONAL AND MEAN-ANNUAL DISPERSIONS OF TURBULENT WIND VELOCITY COMPONENTS, m^2/sec^2

Season	$M[(v')^2]$	$M[(u')^2]$
Winter	73.62	29.57
Spring	56.61	20.96
Summer	23.06	18.63
Autumn	48.68	24.09
Year	50.49	23.32

From Table 2.4 it follows that dispersions of the fluctuations of zonal components exceed by more than a factor of 2 the dispersions in the fluctuations of meridional wind velocity components. An exception is represented by the summer season, when they are approximately the same. Turbulence in this atmospheric layer is most developed in winter. The smallest dispersions are observed in summer.

In Fig. 2.9 is shown the altitude distribution during the year of the kinetic energy per mass of mean and fluctuational horizontal motions. Fig. 2.9 traces two maxima of eddy motions: at an altitude of about 10 km in autumn and at the beginning of winter, and at an altitude of about 46-48 km in winter. The first of these evidently is associated with the tropospheric jet stream. The region of minimum kinetic energies of eddy motion lies in the middle stratosphere, where its center occurs in summer and is observed at an altitude of about 22 km. It indicates the low variability of the easterly wind in summer in the stratosphere.

The kinetic energy of the mean motion has three maxima. The first of these is detected in the stratosphere at an altitude of 12 km and relates to the autumn-winter tropospheric jet stream. The second maximum in autumn and in the beginning of winter lies at the altitude of 50 km. It characterizes the winter jet stream. Finally, the third maximum is noted in summer in the stratosphere.

By comparing the energies of the fluctuational and mean motions, we can see that the largest kinetic energies of eddy motion lie /49 close to the maxima of kinetic energy of the mean motion.

In the mesosphere and the lower part of the thermosphere, turbulence is studied by radar observations of meteor trails. As indicated in Section 2.2, meteors entering the atmosphere of the earth heat up due to friction and vaporize in the 70-110 km layer. As a result, a meteor trail is formed, which is a column of ionized air. The radial electronic density δ_r in the trail is determined by the law

$$\delta_r = \frac{\alpha}{2\pi d_1 t} e^{-\frac{r^2}{4 d_1 t}},$$

in which α is the linear electronic density, t is the time after trail formation, and d_1 is the coefficient of molecular diffusion. This concentration persists during a very short time interval. Later, the meteor trail rapidly deforms under the effect of turbulent eddies. Therefore, its electronic density becomes less. In addition, turbulent eddies produce inhomogeneities of electronic density, which bring about radio wave scattering. As a result, a Doppler frequency shift occurs, whose value is proportional to /50 the radial component of wind velocity. Thus, it now begins possible to determine the horizontal and vertical turbulent pulsations and to estimate how the characteristics of turbulent eddies, for example, their linear and time scales [86, 87].

If it is assumed that the critical Richardson criterion

$$\mathrm{Ri} = \frac{g}{T} \frac{\gamma_a - \gamma}{\left(\frac{dc}{dz}\right)^2}, \qquad (2.19)$$

where c is the wind velocity vector is equal to unity, then

$$\frac{dc}{dz} = \left[\frac{g}{T}(\gamma_a - \gamma)\right]^{1/2}.$$

Since dc/dz has the dimension of angular velocity, we can write

$$\frac{dc}{dz} = \frac{1}{t_1},$$

where t_1 is the characteristic time scale of large eddies. Then Eq. (2.19) becomes

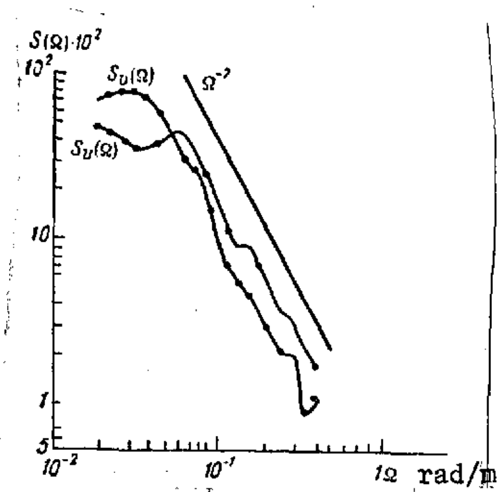


Fig. 2.8. Energy spectra of turbulent component of zonal and meridional wind velocity components

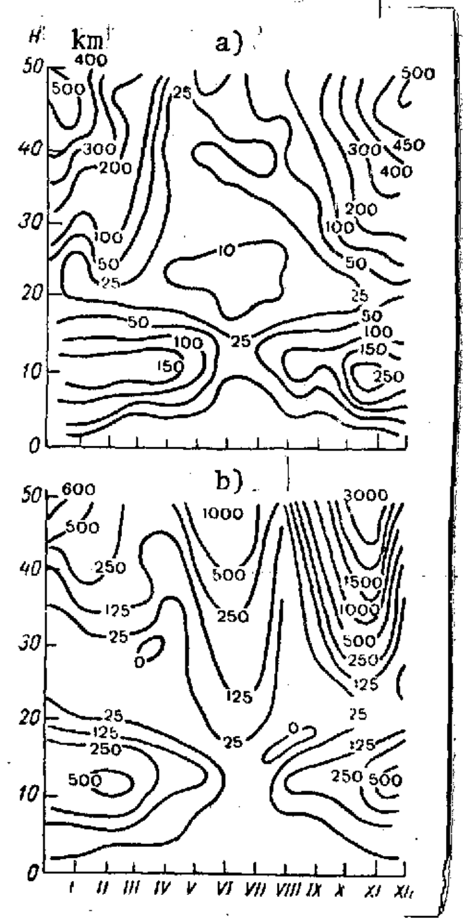


Fig. 2.9. Altitude distribution during the year of the kinetic energy per unit mass of mean (a) and fluctuational (b) horizontal motions

$$t_1 = \left[\frac{g}{T}(\gamma_a - \gamma)\right]^{-1/2} \qquad (2.20)$$

If we denote the specific energy of large eddies per unit time by E_1, the energy during the time t_1 is

$$c_1' = E_1 t_1. \qquad (2.21)$$

where c_1' is the velocity of turbulent pulsations.

The linear scale of large eddies can be obtained by the formula

$$L_1 = c_1' t_1. \qquad (2.22)$$

Eq. (2.11) was used to calculate small dissipating eddies. From this formula it follows that

$$c_2' = (\bar{\varepsilon}\nu)^{1/4} \quad \text{and} \quad t_2 = \left(\frac{\nu}{\bar{\varepsilon}}\right)^{1/2}. \qquad (2.23)$$

Here c_2^1 is the velocity of pulsations in small eddies and t_2 is the characteristic time scale.

Assuming that the velocity of horizontal turbulent pulsations of large eddies is taken as the mean value, 35 m/sec, Booker [87] obtained the following characteristics of large eddies at the altitude 90 km:

$$t_1 = 50 \text{ sec}, \quad E_1 = 25 \text{ W/kg},$$
$$L_1 = 1.6 \text{ km}, \quad dc/dz = 20 \text{ m/(sec}\cdot\text{kg)}.$$

Given the condition $\nu = 4$ m^2/sec, characteristics of dissipating turbulent eddies were also determined, as follows: /51

$$c_2^1 = 3 \text{ m/sec}, \quad t_2 = 0.4 \text{ sec, and } L_2 = \lambda = 1.3 \text{ m}.$$

Greenhow [95-97] made a statistical treatment of radar sounding of 900 meteor trails. He showed that in the 70-110 km layer very large vertical wind shears were observed. In this layer there were wind shears from 0 to 144 m/sec per km of altitude. Their median was 10 m/(sec·km). Wind shears of this value were obtained also by other investigators.

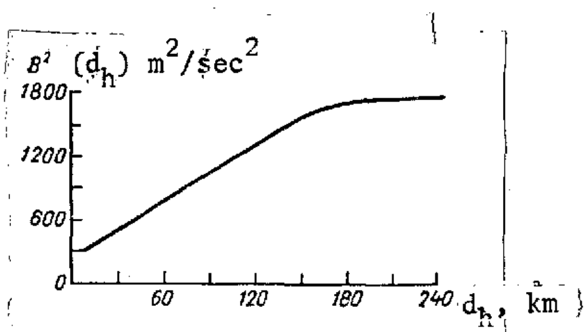


Fig. 2.10. Structure function of horizontal pulsations in wind velocity in the meteor trail layer as a function of horizontal scale of eddies

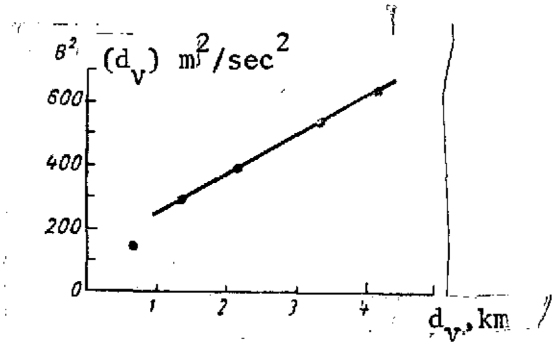


Fig. 2.11. Structure function of horizontal pulsations in wind velocity in the meteor trail layer as a function of vertical scale of eddies

In the meteor trail layer the root mean square values of horizontal turbulent pulsations vary from 15 to 45 m/sec, and their median is 25 m/sec [95-97]. The maximum value of vertical turbulent pulsations does not exceed 10-15 m/sec, with a mean of 2 m/sec.

Horizontal and vertical scales of large eddies can be estimated if we know the correlation functions. The correlation functions of horizontal turbulent pulsations were obtained by Greenhow /95-97/. He established that the horizontal scale of these turbulent pulsations is 150 km or 6000 sec, and the vertical scale somewhat exceeds 7 km. The scale of the lower part of the spectrum of turbulent eddies is about 30 m. Thus, the scales of large and small eddies, as well as their other characteristics are understated by Booker. The above-presented data permit the structure of turbulent motions in the layer of meteor trails to be investigated. Knowing the nature of the correlation functions and the dispersions of the turbulent wind velocities, for example, using Eqs. (2.7) and (2.8) we can calculate the structure functions corresponding to them. Plots of these functions are in Figs. 2.10 and 2.11.

As follows from Fig. 2.10, beginning with the scale of 5 km Yudin's law of "first power" is observed. The structure function reaches saturation approximately at the scale 180 km. This is also confirmed by calculations. If we approximate the structure function with equality (2.14), calculations showed that the dependence of the structure function of horizontal turbulent pulsations on the horizontal eddy scale d_h is of the form /52

$$B_h^2(l_h) = 9d_h, \qquad (2.24)$$

and the dependence of the structure function of horizontal pulsations on the vertical eddy scale d_v is described by the expression

$$B_h^2(l_v) = 120d_v. \qquad (2.25)$$

The structure function of turbulent wind pulsations is porportional to the specific function energy of the turbulent flow. Therefore, if the "first power" law is satisfied, the coefficients in Eqs. (2.24) and (2.25) have the meaning of the gradient of this energy. Their ratio shows by how many times the energy of turbulence changes more rapidly in the vertical direction compared with the horizontal, that is, their ratio characterizes the anisotropy of the turbulent flow. This ratio obviously is equal to 14.

Turbulent eddies with a scale smaller than 5 km obey another structure law. In accordance with the study /70/, the Kolmogorov-Obukhov law of "two-thirds" is this law.

CHAPTER THREE

/53

VERTICAL STATISTICAL STRUCTURE OF PHYSICAL PARAMETERS OF AIR IN DENSE ATMOSPHERIC LAYERS

3.1. Characteristics of Vertical Statistical Structure of Physical Parameters of the Atmosphere

Temperature, pressure, and density of air and winds are random functions of space and time. Therefore a complete description of the structure of fields of these meteorological elements must be based on investigating their space-time statistical characteristics. This investigation is possible when founded on a large series of values of these atmospheric parameters at different points in space and in different time intervals. Unfortunately, quite often this information is lacking. This is particularly true of atmospheric rocket sounding data. Atmospheric rocket sounding at present is the only method by which measurements can be extended to a large fraction of the lower 100-km atmospheric layer. However, rocket sounding is currently carried out from a limited number of proving grounds and at different periods of time. Accordingly, based on rocket sounding data it is possible only to obtain characteristics of the vertical statistical structure of fields of the physical parameter of the atmosphere averaged over large time intervals. In this case, the temperature, pressure, and density of air at some fixed level are viewed as scalar, and the wind velocity values -- as vector random variables.

The fullest characteristic of a random function y is its stochastic description by means of laws of distribution or the functions of the distribution of the densities of probabilities f(y). In several cases, for practical applications it proves to be sufficient to specify random functions and variables by determining simpler characteristics -- the moments of random functions and variables, which are less complete characteristics.

The physical parameters of the atmosphere, such as temperature, /54
pressure, air density, and wind velocity components, are distributed as shown by investigations according to the normal law. Therefore the statistical properties of fields of the physical characteristics are quite fully described by the mathematical expectations

$$M[y] = \int_{-\infty}^{\infty} y f(y)\, dy \tag{3.1}$$

and by the covariances

$$R_y(t_1; t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [y_1 - m_{y_1}][y_2 - m_{y_2}] f(y_1, y_2, t_1, t_2)\, dt_1\, dt_2, \tag{3.2}$$

where t_1 and t_2 are certain arguments.

The covariances are constituent elements of the covariant matrices. Let us examine the covariance matrices of temperature, pressure, and air density as applied to the problem of investigating the vertical statistical structure of the fields of these atmospheric parameters.

The statistical characteristics of the meteorological fields are calculated based on data on temperature $t(H_i)$ and pressure $p(H_i)$ for a series of levels H_i. They make it possible to calculate, using the equation of state (1.7) the density of air at the same levels $\rho(H_i)$. Thus, in the specific case there are values of t, p, and ρ that can be considered as a set of components of the n-dimensional random vector X:

$$X = \begin{Vmatrix} t(H_1) \\ t(H_2) \\ \cdots\cdots \\ t(H_m) \\ p(H_1) \\ p(H_2) \\ \cdots\cdots \\ p(H_m) \\ \rho(H_1) \\ \rho(H_2) \\ \cdots\cdots \\ \rho(H_m) \end{Vmatrix} \quad (n = 3m). \tag{3.3}$$

Let us compare with this vector the vector of mathematical expectations m_X of the same variables. /55

$$m_X = \left\| \begin{matrix} m_t(H_1) \\ m_t(H_2) \\ \dots\dots\dots \\ m_t(H_m) \\ m_p(H_1) \\ m_p(H_2) \\ \dots\dots\dots \\ m_p(H_m) \\ m_\rho(H_1) \\ m_\rho(H_2) \\ \dots\dots\dots \\ m_\rho(H_m) \end{matrix} \right\|. \quad (3.4)$$

Then the generalized covariance matrix of thermodynamic characteristics of atmosphere can be obtained as follows:

$$R_X = M\left[(X - m_X)(X - m_X)^*\right], \quad (3.5)$$

where M is the operation of mathematical expectation, and * is the operation of matrix transposition.

The generalized matrix (3.5) from it it is expanded, is a complex matrix, that can be partitioned into nine blocks:

$$R_X = \left\| \begin{array}{c:c:c} R_{tt} & R_{tp} & R_{t\rho} \\ \hdashline R_{pt} & R_{pp} & R_{p\rho} \\ \hdashline R_{\rho t} & R_{\rho p} & R_{\rho\rho} \end{array} \right\|. \quad (3.6)$$

The blocks lying along the principal diagonal of matrix (3.6) are autocovariance matrices of temperature, pressure, and air density, while the remaining blocks are reciprocal covariance matrices of these meteorological elements. Autocovariance matrices are symmetric matrices, while reciprocal covariance matrices are asymmetric, where

$$R_{pt} = R^*_{tp};\ R_{\rho t} = R^*_{t\rho};\ R_{\rho p} = R^*_{p\rho}.$$

Hence it follows that if we know the blocks lying along the principal diagonal and above it, the generalized matrix (3.6) becomes wholly defined.

Based on matrix 3.6 we can obtain the correlation matrix corres- /56 ponding to it

$$r_X = \left\| \begin{array}{c:c:c} r_{tt} & r_{tp} & r_{t\rho} \\ \hdashline r_{pt} & r_{pp} & r_{p\rho} \\ \hdashline r_{\rho t} & r_{\rho p} & r_{\rho\rho} \end{array} \right\|, \tag{3.7}$$

whose blocks are reciprocal and autocorrelation matrices. There are also asymmetric and symmetric matrices, respectively. The elements of the block matrices comprising the matrices 3.6 and 3.7 are associated by the equalities

$$r_{tt}(H, H') = \frac{R_{tt}(H, H')}{\sqrt{R_{tt}(H, H) R_{tt}(H', H')}}, \tag{3.8}$$

$$r_{tp}(H, H') = \frac{R_{tp}(H, H')}{\sqrt{R_{tt}(H, H) R_{pp}(H', H')}}, \tag{3.9}$$

$$r_{t\rho}(H, H') = \frac{R_{t\rho}(H, H')}{\sqrt{R_{tt}(H, H) R_{\rho\rho}(H', H')}} \tag{3.10}$$

and so on, where the elements under the radical sign are situated along the principal diagonals of the corresponding blocks and constitute dispersions.

The wind factor can be represented as two components, one of which is directed along the parallel, and the second -- along the meridian. As indicated in Chapter Two, these components of wind velocity lie along the axes of a coordinate system, which is taken as standard in meteorology, and the components themselves are called zonal and meridional.

Let us denote the components of the wind velocity vector by u and v, respectively, and let them represent their set in the form of the n-dimensional random vector C.

$$C = \left\| \begin{array}{c} u(H_1) \\ u(H_2) \\ \cdots\cdots \\ u(H_m) \\ v(H_1) \\ v(H_2) \\ \cdots\cdots \\ v(H_m) \end{array} \right\| \quad (n = 2m). \tag{3.11}$$

If we compare the column matrix with the corresponding matrix /57 of mathematical expectations

$$m_C = \left\| \begin{matrix} m_u(H_1) \\ m_u(H_2) \\ \ldots\ldots \\ m_u(H_m) \\ m_v(H_1) \\ m_v(H_2) \\ \ldots\ldots \\ m_v(H_m) \end{matrix} \right\|. \tag{3.12}$$

the covariance matrix of the vector can be obtained as follows:

$$R_C = M\left[(C - m_C)(C - m_C)^*\right]. \tag{3.13}$$

Expanding matrix (3.13), we will get the block matrix

$$R_C = \left\| \begin{array}{c|c} R_{uu} & R_{uv} \\ \hline R_{vu} & R_{vv} \end{array} \right\|. \tag{3.14}$$

The blocks of matrices (3.14) lying along the principal diagonal are symmetric autocovariance matrices of the wind vector components. The remaining two matrices are asymmetric reciprocal covariance matrices and have the property

$$R_{uv} = R^*_{vu}.$$

Matrix (3.14) makes it possible to obtain the correlation matrix

$$r_C = \left\| \begin{array}{c|c} r_{uu} & r_{uv} \\ \hline r_{vu} & r_{vv} \end{array} \right\|, \tag{3.15}$$

which has the same characteristics as the matrix (3.14). Between the elements of the blocks of matrices (3.14) and (3.15) there exists the relation

$$r_{uu}(H, H') = \frac{R_{uu}(H, H')}{\sqrt{R_{uu}(H, H)\, R_{uu}(H', H')}}, \tag{3.16}$$

$$r_{vv}(H, H') = \frac{R_{vv}(H, H')}{\sqrt{R_{vv}(H, H)\, R_{vv}(H', H')}}, \tag{3.17}$$

$$r_{uv}(H, H') = \frac{R_{uv}(H, H')}{\sqrt{R_{uu}(H, H)\, R_{vv}(H', H')}}. \tag{3.18}$$

The dispersions of the wind vector components are under the sign of the radical in expressions (3.16)-(3.18). /58

The elements of the above-considered covariance matrices are calculated from experimental data using the familiar formulas:

$$R_{xy} = \frac{1}{n-1} \sum_{\nu} (x_{\nu} - m_x)(y_{\nu} - m_y), \tag{3.19}$$

$$R_{xx} = D_x = \frac{1}{n-1} \sum_{\nu} (x_{\nu} - m_x)^2, \tag{3.20}$$

$$m_x = \frac{1}{n} \sum_{\nu} x_{\nu}. \tag{3.21}$$

The results of American rocket sounding of the atmosphere in the period 1961 to 1966, pertaining to the US proving grounds listed in Table 3.1, served as the main starting material for calculating the characteristics of the vertical statistical structure of the above-indicated meteorological fields.

TABLE 3.1. STATIONS OF ATMOSPHERIC ROCKET SOUNDING WHOSE DATA UNDERWENT PROCESSING

Station	Latitude	Longitude (West)
White Sands	32°23′ N	106°29′
Point Mugu	34 07	119 07
Cape Kennedy	28 14	80 36
Wollops Island	37 50	75 29
Churchill	58 47	94 17
Fort Greeley	64 00	94 17
Ascension Island	07 59 S	14 28

All the rocket sounding data were divided into two latitudinal groups. The first group included the data of the stations White Sands, Point Mugu, Cape Kennedy, and Wollops Island. The second group comprises the results of rocket launches at the proving grounds of Churchill station and Fort Greeley. In the following we will call these latitudinal groups the middle and high latitudes, respectively. Ascension Island is in the equatorial zone. For the investigation of the vertical statistical structure of the fields of temperature, pressure, and air density, 592 cases were used, and for the investigation of the vertical structure of the wind field -- 1020 cases. These rocket soundings together with data of synchronous radio sounding supplementing them in relation to latitudinal groups and half-year periods into which the starting data were subdivided dealt with the layer up to 50-70 km. The period of time from April to October refers to the cold half-year, and from October to April refers to the warm half-year. /59

The rocket sounding data used for the statistical interpretation as a rule were more than two days apart. To investigate the diurnal variability of the air density, a number of observations taken a day apart were examined. These data numbered 72 in the warm period, and 82 in the cold.

The precision of the measurements of atmospheric parameters was characterized by the root mean square error of the individual measurements. The meteorological rockets took measurements of the meteorological elements with the following precision: the root mean square error of the pressure measurements was 4%; the root mean square error of the temperature measurements was 2-3°C. As a result, the air density based on rocket sounding was determined to an average position of 5% [90]. The root mean square error of wind velocity measurement in the atmospheric layer 10-70 km was 0.5-2.5 m/sec, if the wind measurements were made by means of a parachute device and falling spheres, and about 10 m/sec when the wind was measured by radar observations behind the clouds of metallized dipoles [105, 107, 108].

3.2. Altitude Distribution of Mean Values of Temperature, Pressure, and Air Density in the Dense Atmospheric Layers

To expand the characteristics of the vertical statistical structure of the fields of temperature, pressure, and air density, let us consider the altitude distribution of several statistical characteristics of these physical parameters of the atmosphere in different latitudinal zones and half-years over the North American continent. A comparison of these characteristics with those obtained for other regions in the northern hemisphere shows that the general regularities of the vertical structure have no essential differences, although there are some differences in details.

The altitude distribution of air temperature averaged by half-years and latitudinal zones is shown in Fig. 3.1 for the North American continent; from this figure it follows that the vertical profile of the mean temperature in the middle latitudes differ to a considerable extent from the vertical profiles of the mean temperature of the upper latitudes.

The profiles of the half-year averaged temperature agree closely with familiar concepts of the distribution of air temperature in the dense atmospheric layers, which are briefly expounded in Chapter One, concerning the difference between the vertical and horizontal gradients of temperature in the cold and warm half-years.

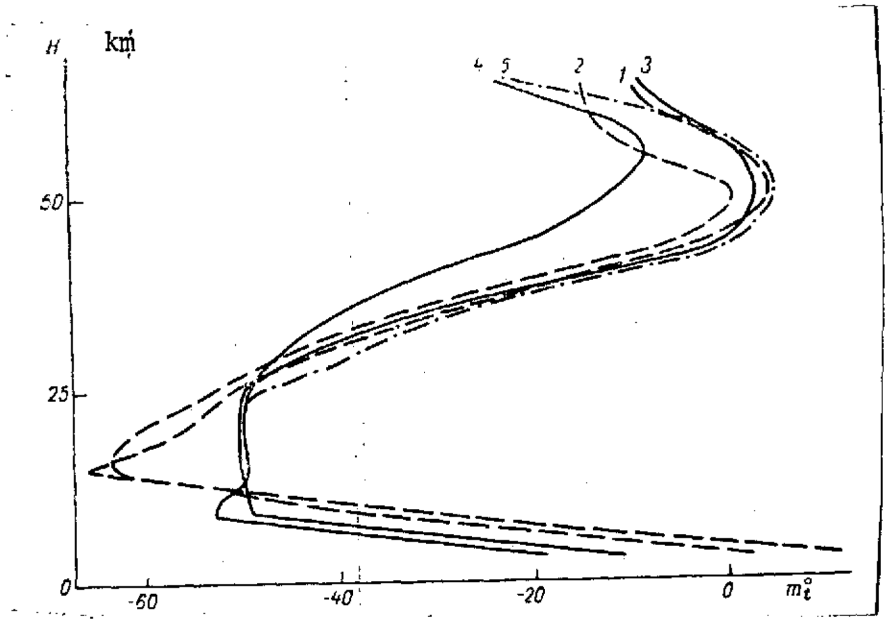


Fig. 3.1. Altitude distribution of air temperature averaged by half-years and latitudinal zones:
Middle latitudes: warm period (1) and cold period period (2)
Northern latitudes: warm period (3) and cold period (4)
Equatorial zone (5)

In addition to the mean temperature, of high interest are data characterizing fluctuations in air temperature at different altitudes. Fig. 3.2 a shows the deviation from the mean value of extremal temperatures in different latitudinal zones and different half-years observed in the period 1961 to 1966. From Fig. 3.2 a it follows that in the stratosphere and lower mesosphere the temperature can differ 20-45°C from the mean value. The largest positive temperature deviations occur above 20 km in the cold half-year. /61

They occur in the winter months and are associated, as already noted above, with abrupt warmings in the stratosphere.

Large fluctuations in temperature in these atmospheric layers are indicated by the fairly high values of the root mean square deviations of temperature, shown in Fig. 3.2 b. From this figure it follows that the largest root mean square deviations of temperature observed in the cold half-year in the northern latitudes in the 30-60 km layer. Above 50-60 km, the root mean square deviations of temperature in the northern latitudes decrease sharply, while in the middle latitudes they remain nearly unchanged.

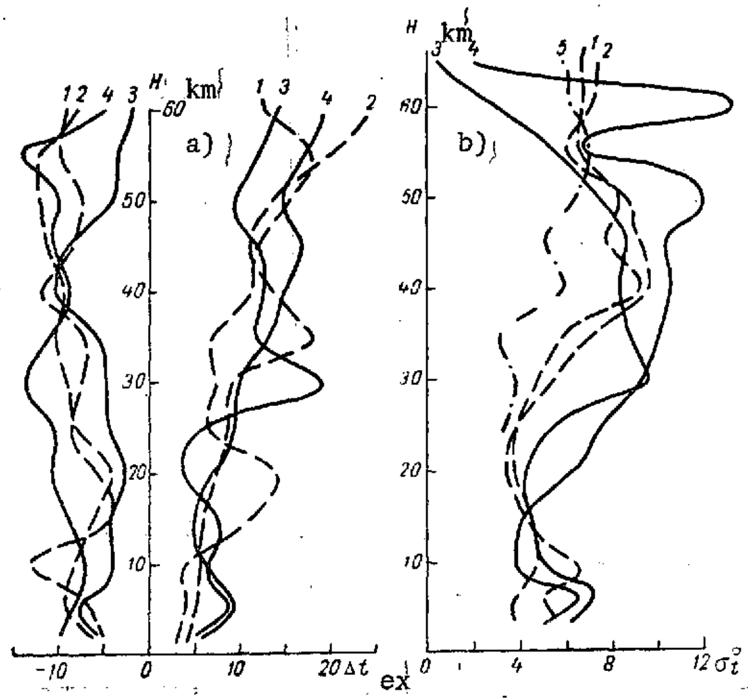


Fig. 3.2. Deviations from the mean value of extremal temperatures (a) and root mean square deviations from temperature (b). The symbols are as in Fig. 3.1.

Fig. 3.3 presents the deviations of the mean temperature values from the temperature based on the 1964 standard atmosphere (SA-64). Fig. 3.3 shows that the mean temperature values as a function of region and half-year can differ widely from the standard values. In the troposphere, the mean temperature is lower than the standard in the upper latitudes, where the difference between them in the cold-half year based on absolute magnitude at the altitude 3 km exceeds 20°C, while the mean temperatures above the standard in the middle altitudes. At the altitude 12 km the mean temperature in both latitudinal zones and half-year is nearly identical and is equal to the standard value. In the 12-25 km atmospheric layer in the upper latitudes the stratosphere proves to be warmer than the standard value. But in the cold half-year the mean air temperature is below the standard value. Only in the lower mesosphere are positive deviations of temperature from the standard value observed, and they increase sharply with altitude, reaching 27°C at the altitude 65 km. /62

In the cold half-year on the average at all altitudes the atmosphere in the higher latitudes is colder than the standard value. The maximum value of the negative temperature deviations occurs at the altitude 45 km and is 17°C.

Table 3.2 presents the mean air temperatures and the temperature based on the CIRA-1965 Standard Atmosphere. The mean temperature values are taken from collections of atmospheric rocket sounding data /120/. They were calculated for the period 1961 to 1965.

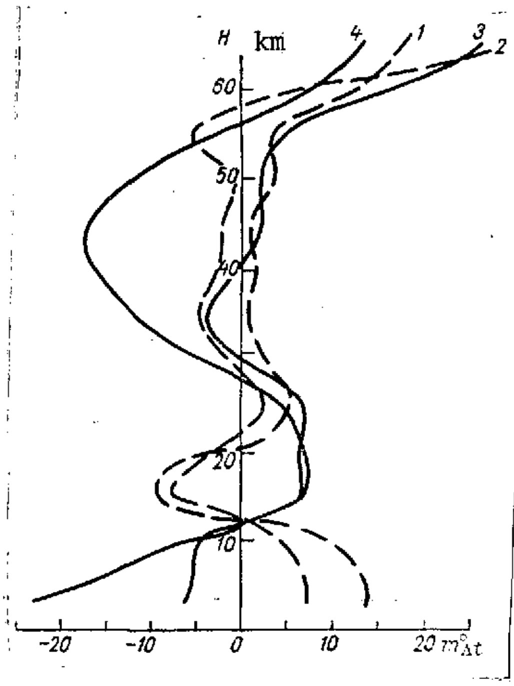


Fig. 3.3. Deviations of mean air temperature from standard value:
Symbols as in Fig. 3.1.

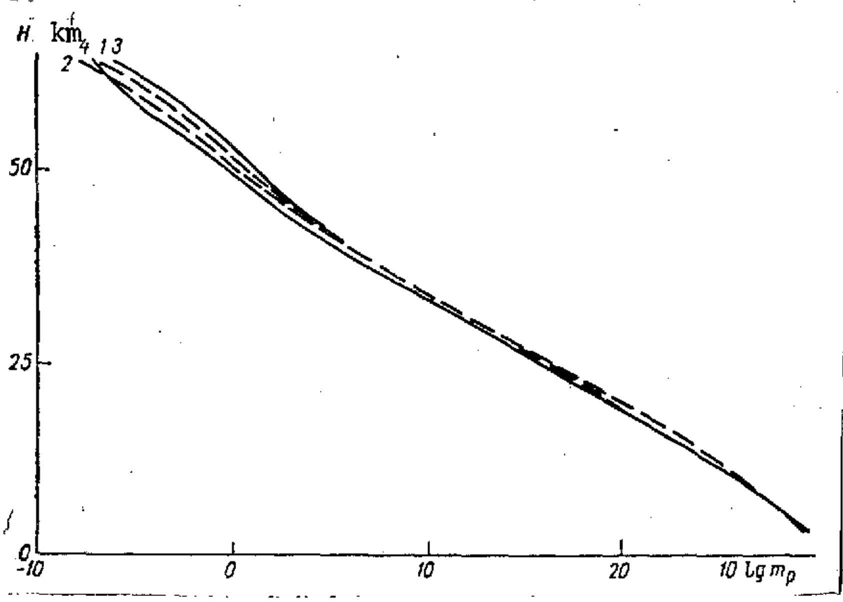
Analysis of the data in Table 3.2 shows that above 30 km there most of the levels the mean temperatures for both stations prove to be lower than the temperature based on CIRA-1965. An exception is represented by the temperature in May at the White Sands station and also in May and October at the altitudes 60 km in the station Fort Greeley.

The mean atmospheric pressure depends to a large extent on the time of the year and the latitude. We compare the altitude distributions of the mean pressures for the warm and cold half-year (Fig. 3.4), we can conclude that the largest values of the mean pressure are observed in the zone 30-40° N. Lat in the warm half-year, and the smallest -- in the upper latitudes (50-60° N. Lat) in the cold half-year. The difference between

TABLE 3.2. MEAN VALUES OF TEMPERATURE $\bar{t}$ AND TEMPERATURE BASED ON CIRA-1965 (°C)

H km	White Sands				Fort Greeley			
	May		October		May		October	
	$\bar{t}$	CIRA-1965	$\bar{t}$	CIRA-1965	$\bar{t}$	CIRA-1965	$\bar{t}$	CIRA-1965
30	−41.1	−39.0	−45.8	−40.0	−45.3	−50.0	−51.5	−49.0
40	−12.6	−16.0	−23.5	−23.5	−13.4	−19.0	−36.3	−27.0
50	4.9	− 2.0	−41	1.0	2.9	11.0	−13.4	− 4.0
60	− 6.1	−21.0	−11.7	− 4.0	− 4.4	−18.0	−11.8	−31.0

them at the altitude 3 km is 28 mb, at the altitude 6 km -- 31 mb, and at the altitude 9 km -- 21 mb. These differences are due to the predominance in the cold period of all anticyclonic regime in the mid-latitude troposphere, and the predominance of intense cyclonic activity in the upper latitudes during the cold period. This nature of the differences in the mean pressure values by half-year and latitudinal zones persists up to altitudes of about 40 km. Above 45 km, the largest mean pressure is observed in the warm half-year in the upper latitudes.



/63

Fig. 3.4. Distribution of mean air pressure by altitude, half-years, and longitudinal zones:
Middle latitudes: warm period (1) and cold period (2); northern latitudes: warm period (3) and cold period (4)

Fig. 3.5 gives the results of comparing the mean air pressure values with the pressure based on the SA-64 Standard Atmosphere. As follows from Fig. 3.5, up to the altitude of 40 km in the mid-latitudes, the mean pressure somewhat exceeds the standard pressure. An exception is represented by a narrow layer near the 20 km level. /64 Above 40 km the mean pressure in the cold half-year in the mid-latitudes becomes less than the standard. In the warm half-year at these latitudes the mean pressure remains higher than the standard, where above 50 km the difference between them grows. In the high latitudes in the warm half-year the mean pressure up to the altitude 45 km is 5-7% smaller than the standard. Above 45 km the difference between them becomes positive and rises sharply with altitude, exceeding 40% at the altitude 60 km.

In the cold half-year the mean pressure in all of the atmospheric layer considered is below the standard value in the upper latitudes. With increase in altitude, the difference between the values steadily rises, reaching 20% at the altitude 55 km.

Table 3.3 presents the mean pressure values as well as the pressures from the CIRA-1965 model.

Data given in Table 3.3 showed that in the warm time of the year the mean pressure at the stations exceeds the pressure based on the CIRA-1965 Standard Atmosphere. In the cold period the opposite picture is observed. Particularly large differences occur at the high latitudes.

The possible limits to the variability of air pressure at different altitudes can be judged from the data given in Fig. 3.6; here are shown the relative deviations of the extremal pressure values observed over the North American continent in the period 1961-1966 of the mean value. From Fig. 3.6 it follows that in the high latitudes there are greater extremal deviations than in

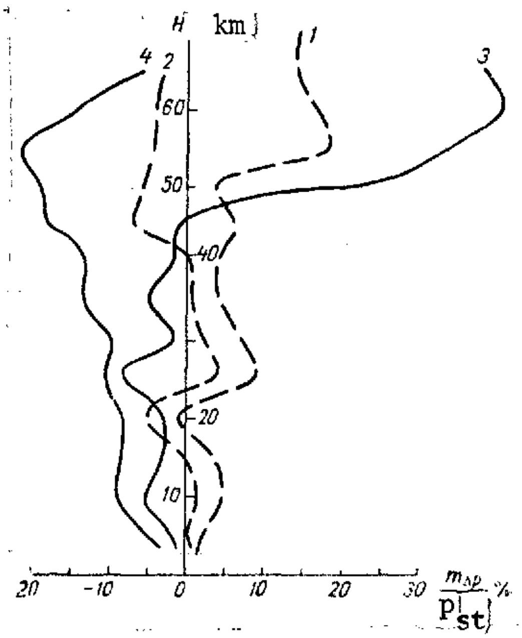


Fig. 3.5. Deviation of mean air pressure from standard value: Symbols as in Fig. 3.4

TABLE 3.3. MEAN PRESSURE VALUE $\bar{p}$ AND PRESSURE ACCORDING TO CIRA-1965 (mb)

H, km	White Sands				Fort Greeley			
	May		October		May		October	
	$\bar{p}$	CIRA-1965	$\bar{p}$	CIRA-1965	$\bar{p}$	CIRA-1965	$\bar{p}$	CIRA-1965
30	12.541	12.100	12.186	12.500	12.550	11.300	11.509	12.900
35	6.162	6.000	5.877	6.150	6.060	5.300	5.384	6.310
40	3.189	3.030	2.919	3.120	3.110	2.620	2.607	3.200
45	1.701	1.570	1.514	1.650	1.660	1.350	1.302	1.670
50	0.962	0.830	0.817	0.887	1.060	0.720	0.685	0.896
55	0.540	0.445	0.437	0.476	0.630	0.380	0.356	0.482
60	0.281	0.230	0.230	0.248	0.341	0.196	0.206	0.254

the middle latitudes. The largest values are noted in the cold half-year period. In the stratosphere two maxima of deviations are detected, one of which lies in the 25-35 km layer, and the other lies in the region of the stratopause. These maxima are equal to 50 and 60%, respectively.

The curves corresponding to the negative extreme of deviations of pressure from the mean do not exhibit so well-defined differences. However, we can trace the maximum in the mid-latitudes at the altitude of about 40 km, and in the upper latitudes at the altitude of about 50 km. The data in Fig. 3.6 further indicate that in the stratosphere and lower mesosphere the pressure can vary within appreciable limits. These changes can attain 80-100% in the stratosphere and 100-120% in the mesosphere. /65

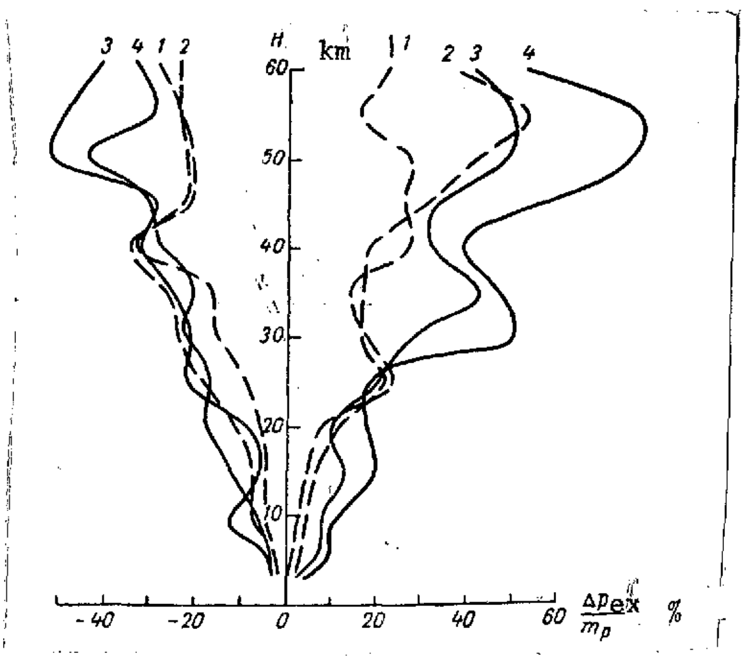


Fig. 3.6. Deviations in extremal pressure values from the mean. Symbols as in Fig. 3.4.

The large changes in air pressure in these atmospheric layers are indicated by the root mean square deviations of pressure given in Fig. 3.7 in ratios to the mean pressure at the corresponding altitude. From Fig. 3.7 it follows that the root mean square deviations of pressure in the upper latitudes have a larger value than in the middle latitudes. At the curve shown in Fig. 3.7 two maxima are clearly evident, which lie in the 30-40 and 50-60 km layers, that is, they coincide with the maxima at the curves of the extremal deviations. The root mean square deviations here are 15-20 and 20-30%, respectively.

The general pattern of variation in the mean air density with altitude as a function of half-year and latitude is similar to the pattern of change in pressure. This is evident if we compare Figs. 3.4 and 3.8. In Fig. 3.8 is shown the variation in mean density with altitude above the North American continent. However, the curves in Figs. 3.4 and 3.8 also have appreciable differences. The most important difference is a large mean air density in the high-latitude troposphere compared to the mean density of air in this atmospheric layer in the middle latitudes. /66

These data indicate that the smallest mean air density in the stratosphere and lower mesosphere occurs in winter and the northern latitudes. The highest mean air density in the stratosphere is observed during the warm half-year in the middle latitudes. This phenomenon can be traced up to an altitude of approximately 40 km, that is, just as for air pressure.

The density of the atmosphere can undergo appreciable fluctuations about the mean. This is clearly seen in Fig. 3.9, where the relative deviations of the extremal air density is observed in the period of 1961 to 1966 from the mean values are presented. They are especially large in the high-latitude stratosphere in the cold half-year, although in the middle latitudes even in the warm half-year at some levels they exceed 20% in the upper latitudes. The curves of the extremal deviations have two well-defined maxima, one of which is at the altitudes 25-30 km, and the second -- at the altitudes 45-50 km. These maxima are especially large in the northern latitudes in the cold half-year. At the altitude 30 km the density of the air can exceed the mean by 60%, and at the altitude 45 km -- by approximately 70%. Therefore, if we consider that the deviation of the minimum air density is from the mean, the density in the stratosphere can vary by 50-70%, and in some altitudes -- by 100% or more.

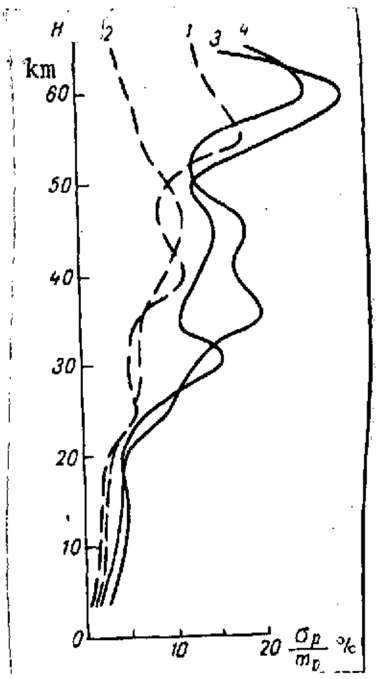


Fig. 3.7. Root mean square deviation of air pressure
Symbols as in Fig. 3.4

The density of air in the stratosphere can vary quite rapidly. In Fig. 3.10 is presented the altitude distribution of the root mean square changes in air density (square roots of the time-based structure functions) over a 24-hour time interval. Fig. 3.10 shows that in the course of a 24-hour day in the mid-latitude stratosphere the air density has a variability of 7-12%, while in the upper-latitude stratosphere -- 10-17%.

The high variability in air density in this atmospheric layer is indicated by the root mean square values of deviations, shown in Fig. 3.11. From the /68
data presented, it can be concluded that the root mean square deviations of air density in the troposphere are relatively small and amount to 2-5%. With increase in altitude, they climb, exceeding 10-15% in the stratosphere and 20-25% in the lower mesosphere.

The nature of the distribution of air densities in the 70-80 km layer can be judged from the data in Table 2.4. The values given there were obtained by averaging air density data for two groups of latitudes in the North American continent.

/67

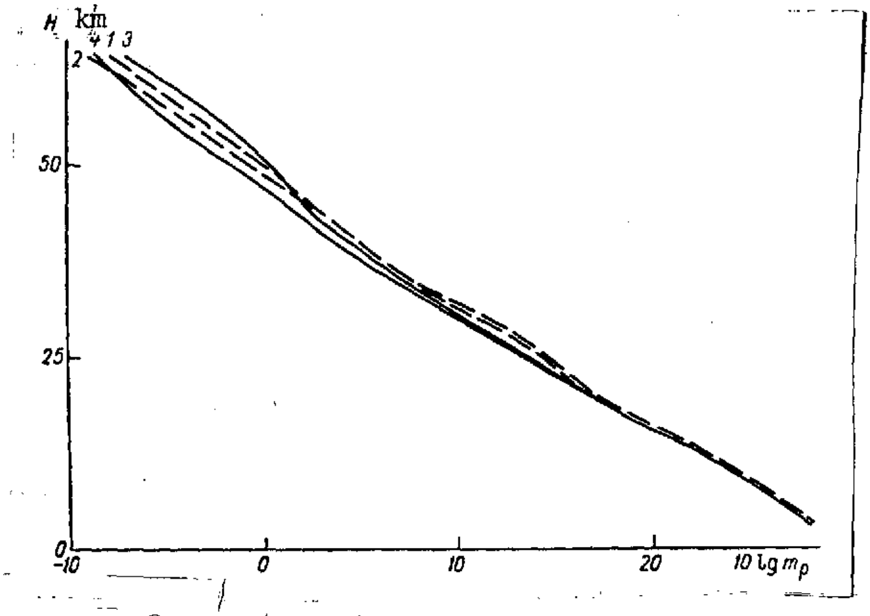


Fig. 3.8. Distribution of air density by altitude, half-year, and latitudinal zone
Middle latitudes: warm period (1) and cold period (2); northern latitudes: warm period (3) and cold period (4)

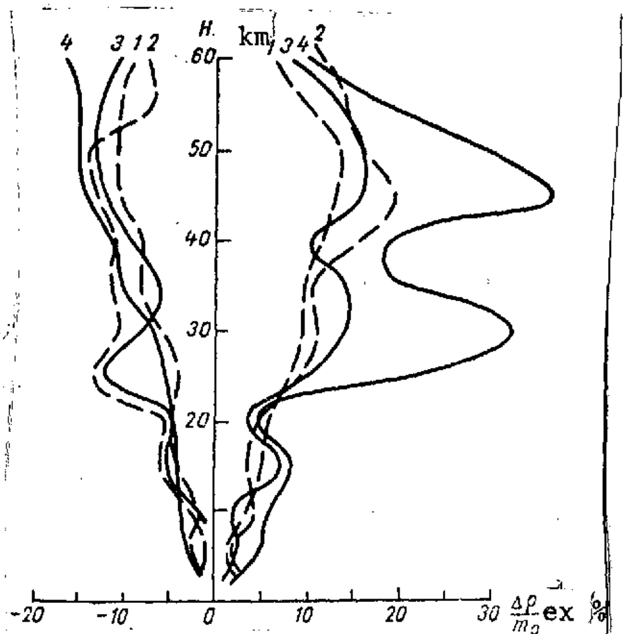


Fig. 3.9. Deviations of extremal air densities from mean
Symbols as in Fig. 3.8.

From Table 3.4 it follows /69
that at the altitude 70-80 km the difference between the maximum and the minimum values completely proves to be larger than the mean density as such.

In the period 1963 to 1964 13 atmospheric soundings were conducted over Kwajalein Island (9° 24' N. Lat; 167° 39' E. Long) using meteorological rockets, which reach the altitudes of 100 km. From the data of these ascents, mean values were obtained for air density, as well as extremal and root mean square deviations. These values can naturally aspire only to the rule of approximate estimated characteristics (Table 3.5).

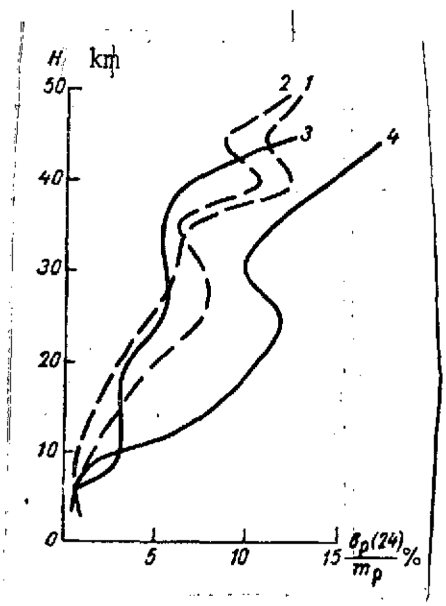


Fig. 3.10. Variability in air density in a 24-hr interval. Symbols: see Fig. 3.8

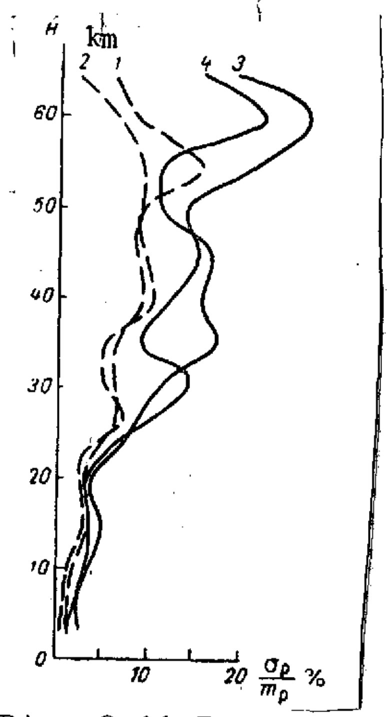


Fig 3.11 Root mean square deviations of air density. Symbols: see Fig. 3.8.

TABLE 3.4. MEAN EXTREMAL VALUES AND ROOT MEAN SQUARE DEVIATIONS OF AIR DENSITY (g/m^3)

H km	No. of meas.	$\bar{\rho}$	ρ_{min}	ρ_{max}	σ_ρ	$\frac{\sigma_\rho}{\bar{\rho}}$ %
70	52	0.126	0.055	0.260	0.024	19.1
80	25	0.0196	0.0106	0.030	0.006	30.3

TABLE 3.5 HEIGHTWISE DISTRIBUTION OF AIR DENSITY IN 80-100 KM LAYER FROM DATA OF 13 SOUNDINGS

H km	$\bar{\rho}$ g/m^3	$\frac{\rho_{max}-\bar{\rho}}{\bar{\rho}}$ %	$\frac{\rho_{min}-\bar{\rho}}{\bar{\rho}}$ %	σ_ρ g/m^3	$\frac{\sigma_\rho}{\bar{\rho}}$ %
85	0.00836	20.0	−14.7	0.00074	8.9
90	0.00374	9.2	−10.1	0.00023	6.6
95	0.00158	30.0	−31.0	0.00025	15.8
100	0.00053	17.1	−17.0	0.00006	10.6

TABLE 3.6. DENSITY VALUES FROM ROCKET SOUNDINGS AND METEOR PHOTOGRAPHY (g/m^3)

H km	$\bar{\rho}$ from rocket soundings	$\bar{\rho}$ from meteors
85	0.00836	0.00912
90	0.00374	0.00357
95	0.00158	0.00112
100	0.00053	0.00054

Table 3.6 presents for comparison atmospheric density values from rocket soundings over Kwajalein Island and by photographing meteors over Kiev (see Section 1.3).

The values in Table 3.6 show that the densities from rocket soundings and meteor photography agree well with each other.

Of interest are the results of comparing the mean density of air with the standard density based on the SA-64 standard atmosphere, shown in Fig. 3.1. In the middle latitudes up to an altitude of 10 km, the mean air density proves to be lower than the standard. Above this level up to the altitude of approximately 35 km it exceeds the standard in both half-years. Above 35 km the mean density remains larger than the standard in the warm half-year. The maximum difference between them occurs at the altitude 55 km and is 13%. In the cold half of the year above 35 km the mean air density becomes less than the standard. With increase in altitude, the difference between them climbs, reaching 12% at the altitudes 60 km. /70

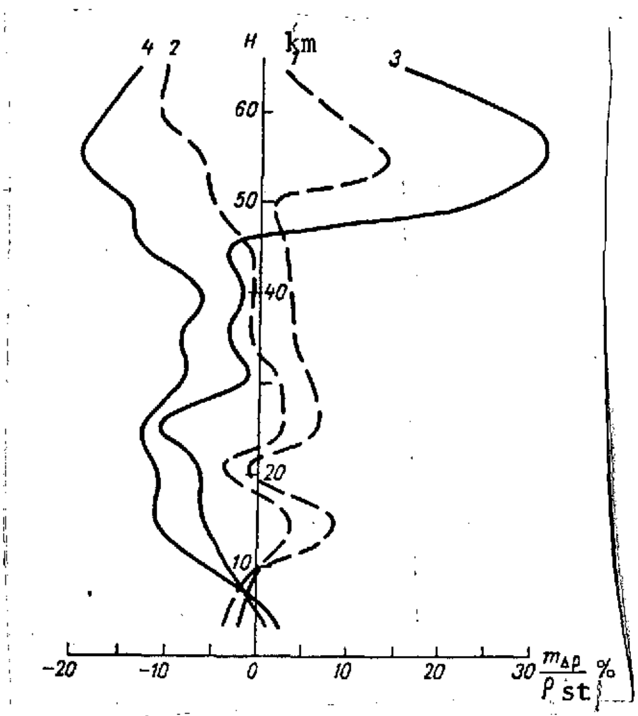


Fig. 3.12. Deviations of mean air density from the standard value Symbols as in Fig. 3.8.

In the high latitudes the mean air density is larger than the standard value in the lowest tropospheric layer. Above 5-7 km and up to the altitude of 45 km, the mean air density in the cold and warm half-years proved to be less than the standard. Above 45 km in the warm half of the year the mean density again becomes larger than the standard and the difference between them rises sharply with altitude. At an altitude of about 55 km it is already 30% higher than the density based on the standard atmosphere. In the cold half-year the mean air density above 45 km in the high latitudes remains less than the standard. With increase in altitude, the difference between them rises, reaching a maximum of 20% of the altitude 55 km.

Table 3.7 gives the deviations of the mean air density from the standard based on sounding data over Kwajalein Island.

The data given above characterize the deviations of the mean values of air density from the standard value. However, it is of interest to find how often particular deviations from standard values can be observed. An idea of this can be afforded by

TABLE 3.7. DEVIATIONS OF AIR DENSITY FROM STANDARD VALUE OVER KWAJALEIN ISLAND

/71

H km	70	75	80	85	90	95	100
$\frac{\rho-\rho_{st}}{\rho_{st}}$ %	35.9	10.5	−6.7	−2.0	7.7	4.3	−2.2

histograms of the deviations of air density in the stratosphere and lower mesosphere. In Fig. 3.13 are shown histograms of the relative deviations of air density for the warm half-year in the middle latitudes. They indicate that at the altitude of 20 km the deviations from 0 to -10% have the highest incidence. Beginning at the altitude 25 km the maximum incidence shifts to the gradations 0 to 10%. Up to the altitude of 40 km it is approximately 60%. Above 40 km the incidence of the deviations exceeding 10% in absolute value rise sharply. Histograms for cold half-year in the middle latitudes and warm half-year in the upper latitudes are of similar form.

Altogether different is the pattern of the histograms for the high-latitude cold half-year (Fig. 3.14). The difference between them is that at these levels the center of distribution shifts toward the region of negative gradations and does so more strongly, the higher the level. This indicates that in the vast majority of cases in the cold half-year there are negative deviations of air density from standard values. Quite often cases are encountered when the air density proves to be 40, 50, and even 60% lower than the standard value.

Table 3.8 presents for comparison the mean air densities over the White Sands and Fort Greeley stations and the air densities based on the CIRA-1965 Standard Atmosphere.

TABLE 3.8. MEAN VALUE OF DENSITY $\bar{\rho}$ AND AIR DENSITY FROM CIRA-1965 (g/m3)

H km	White Sands				Fort Greeley			
	May		October		May		October	
	$\bar{\rho}$	CIRA-1965	$\bar{\rho}$	CIRA-1965	$\bar{\rho}$	CIRA-1965	$\bar{\rho}$	CIRA-1965
30	18.779	18.500	18.761	18.700	19.092	18.900	18.032	19.700
35	8.772	8.680	8.738	8.800	8.683	8.460	8.246	8.540
40	4.264	4.250	4.073	4.250	4.129	3.980	3.846	4.020
45	2.166	2.160	1.989	2.130	2.116	1.980	1.803	1.904
50	1.203	1.114	1.055	1.130	1.327	1.080	0.917	1.010
55	0.671	0.622	0.573	0.627	0.790	0.615	0.470	0.559
60	0.353	0.342	0.307	0.346	0.428	0.347	0.274	0.299

/72

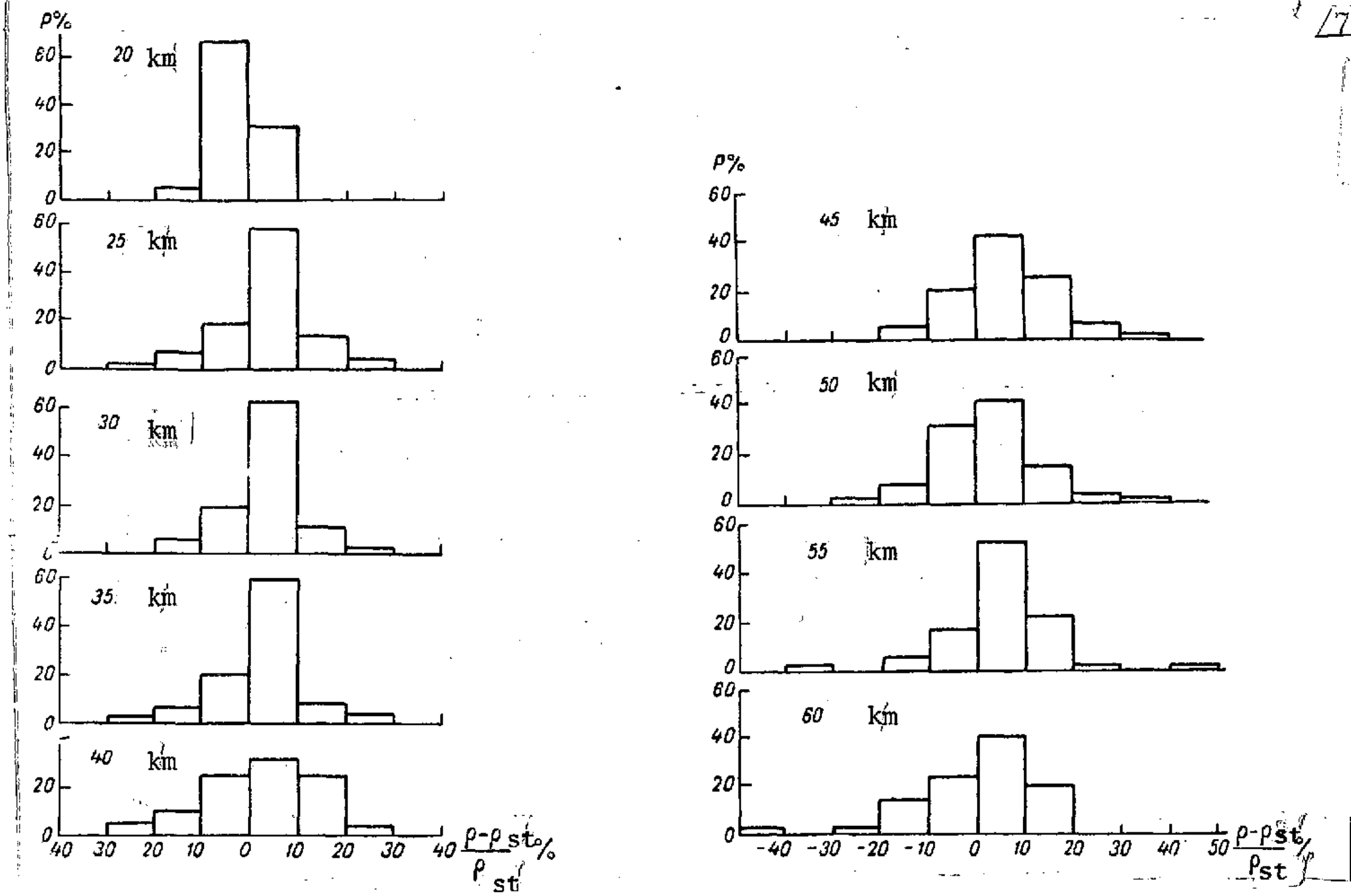


Fig. 3.13. Histogram of deviations of air density from standard. Warm half-year, middle latitudes

From Table 3.8 it follows that over both these stations in May, the mean density exceeds the CIRA-1965 density, and does so by a fairly large value in the high latitudes. In October the opposite picture is observed. Both in May and October, the greatest differences between these densities are observed in the high latitudes. /74

These results of a comparison of temperature, pressure, and air density with the SIA-64 and the CIRA-1965 models show that even the space-time model of the CIRA-1965 atmosphere that is more physically substantiated does not adequately reflect the actual distribution of the physical parameters of the atmosphere.

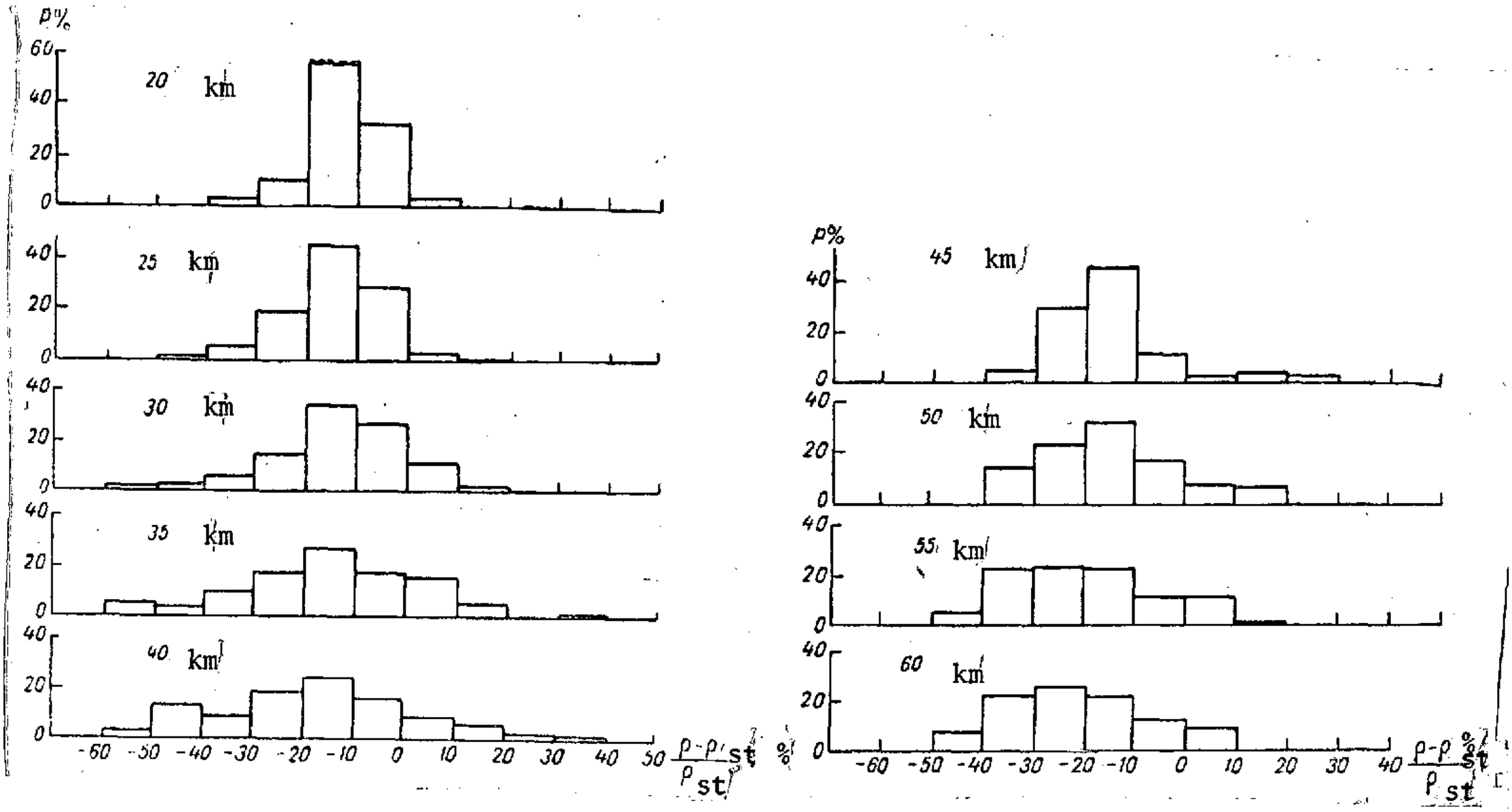


Fig. 3.14. Histogram of deviations of air density from standard value. Cold half-year, high latitudes

3.3. Correlation Matrices of Temperature, Pressure, and Air Density

The first block of the principal diagonal of the generalized covariant matrix (Fig. 3.6) is an autocovariance matrix of air temperature. It is a symmetric matrix, along whose principal diagonal allocated the values of the temperature dispersions at altitudes. To this matrix there corresponds the autocorrelation matrix of temperature.

Autocorrelation matrices are conveniently represented for purposes of analysis in the form of autocorrelation functions.

Autocorrelation functions of temperature have several characteristic features. They are shown in Figs. 3.15 and 3.16 for the middle and high latitudes of North America. The numbers alongside the curves in the figures denote the altitude in kilometers of the initial correlation level.

Autocorrelation functions of temperature for the middle latitudes whose initial correlation levels lie in the troposphere (in the following we will call these functions tropospheric, in contrast to stratospheric functions, that is, functions with the initial correlation levels located in the stratosphere) have approximately identical form in the warm (Fig. 3.15 b) and cold (Fig. 3.15 a) half-year. They are characterized by the fact that with increase in altitude the correlation in the troposphere decreases rapidly, reaching zero at the altitude 10-12 km. Above this level the correlation becomes reciprocal. The maximum reciprocal relation is observed at the altitude 15 km. These values of the autocorrelation function indicate that a rise in temperature at the altitude 15 km usually accompanies its decrease at all levels.

Stratospheric functions in the cold period and functions with initial correlation levels of 15 and 20 km in the warm period are identical in form and indicate a fairly rapid decrease in the correlation with altitude. Stratospheric functions with initial correlation levels above 20 km in the warm half-year decrease more slowly with increase in altitude.

In the high latitudes, the autocorrelation functions of temperature corresponding to the 3 and 6 km initial correlation levels also decrease rapidly with altitude, however, the reciprocal correlation, whose maximum as in the middle latitudes occurs at the altitude 15 km has a smaller value. The pattern of the correlation of temperature in the stratosphere in the cold (Fig. 3.16 a) and warm (Fig. 3.16 b) half-years in the high latitudes is identical and is similar to the pattern of the correlation in the warm period in the middle latitudes of above 20 km. /75

Thus, in the graphs of the autocorrelation temperature function we can distinguish four groups of curves. The first and the second groups refer to the initial levels of correlation located in the troposphere for the middle and high latitudes, respectively. The third group can encompass the stratospheric curves for the cold half-year and functions with initial levels of 15 and 20 km for the mid-latitude warm half-year. Finally, the fourth group includes the autocorrelation functions of temperature for the mid-latitude warm half-year with initial correlation levels of above 20 km, and also for the high-latitude warm and cold half-year with initial correlation levels located in the atmosphere. /77

Autocorrelation matrices of pressure, as the above-described matrices of temperature, are also conveniently represented as autocorrelation functions. These functions have several features. In the middle latitudes in the cold half-year (Fig. 3.17), the autocorrelation functions of pressure for which the initial correlation levels are in the troposphere decrease rapidly. Above 25 km they again begin to rise somewhat. Stratospheric autocorrelation functions beginning at the altitude 30 km indicate the smoother

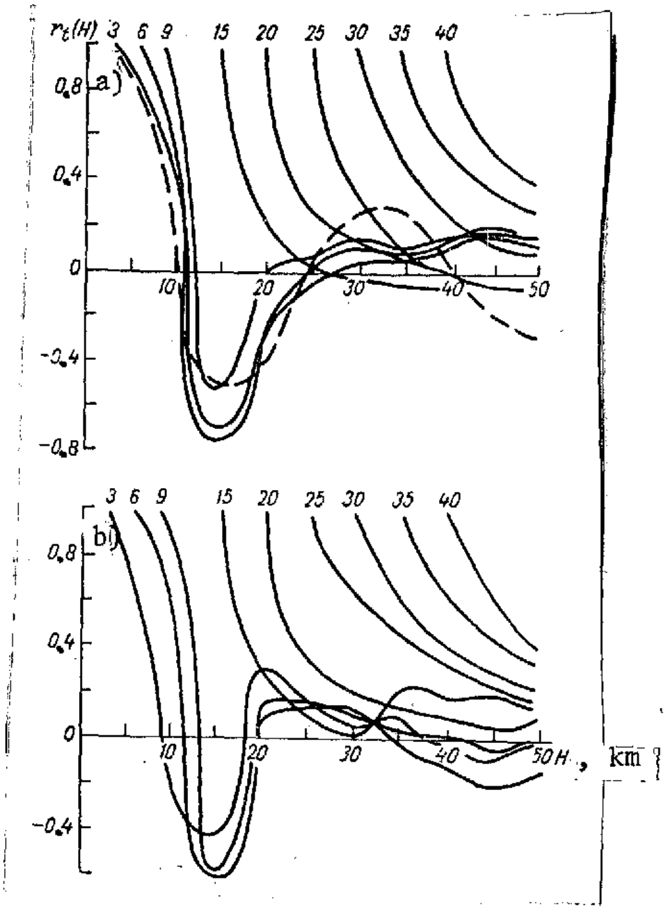


Fig. 3.15. Correlation functions of temperature for the cold (a) and warm (b) half-years in the middle latitudes

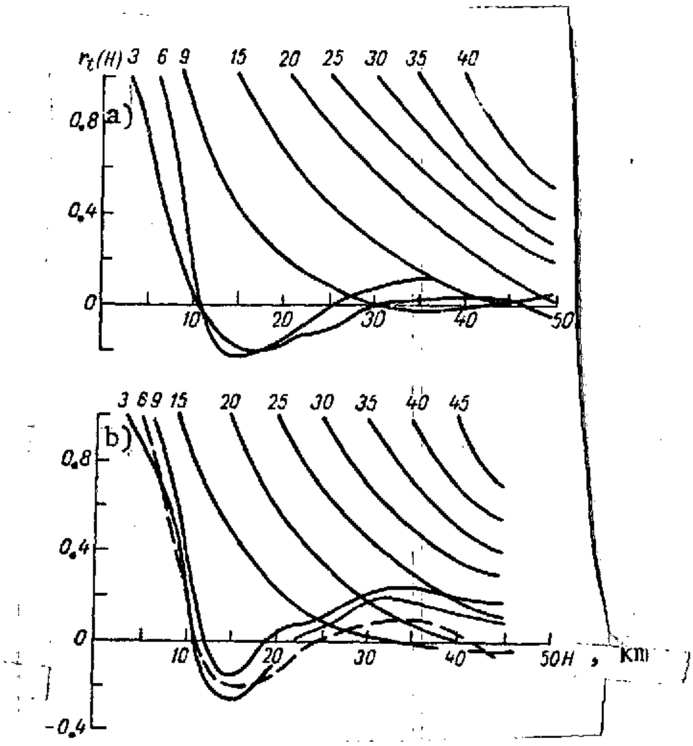


Fig. 3.16. Correlation functions of temperature for the cold (a) and warm (b) half-years of the high latitudes

decrease in the correlation of pressure at the lower-lying levels with pressure at the high altitudes. In the high latitudes the nature of the correlation of pressure and different levels is approximately the same.

Autocorrelation matrices of air density are also, as is true of autocorrelation matrices of temperature and pressure, the corresponding blocks of the generalized matrix (3.7). Figs. 3.18 and 3.19 show the autocorrelation matrices of air density for the middle and high latitudes in the form of autocorrelation functions. As before, these numbers alongside the curves denote altitudes in kilometers of the initial correlation levels. /78

/76

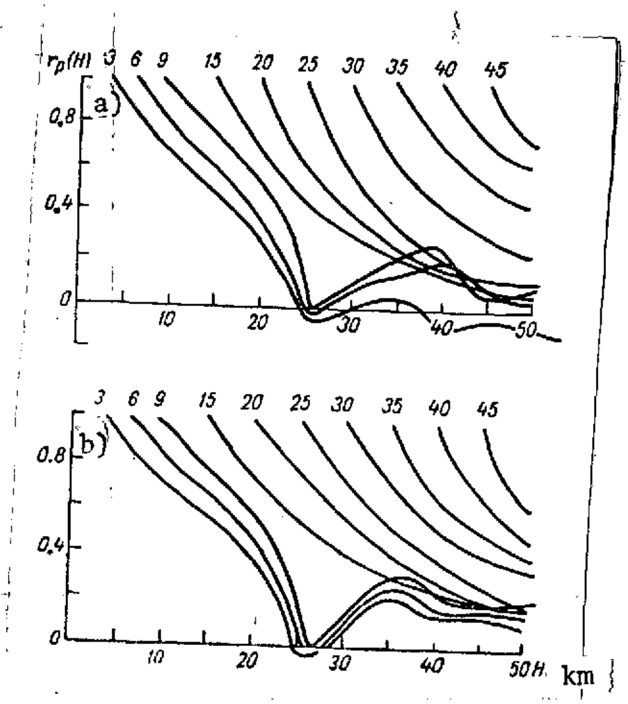


Fig. 3.17. Correlation functions of pressure for the cold (a) and warm (b) half-years of the middle latitudes

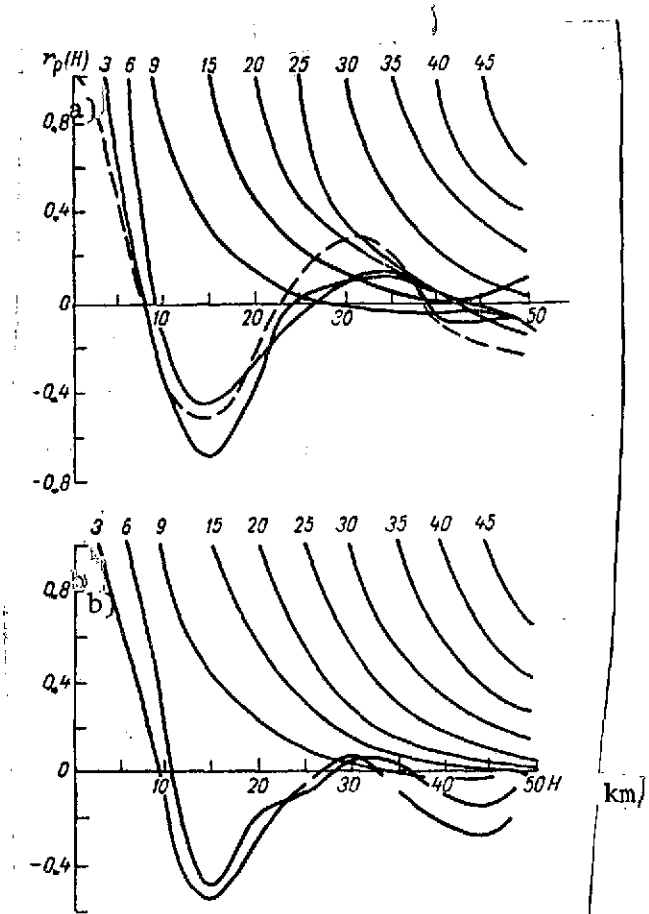


Fig. 3.18. Correlation functions of air density for the cold (a) and warm (b) mid-latitude half-years

In the middle latitudes in the cold (Fig. 3.18 a) and warm (Fig. 3.18 b) half-years, functions with 3 and 6 km initial correlation levels decrease rapidly with increase in altitude, reaching zero near the altitude of 10 km, and then take on a negative value of 0.5-0.7 at the altitude 15 km. Corresponding functions belonging to the cold (Fig. 3.19 a) and warm (Fig. 3.19 b) high-latitude half-year differ from the above-indicated functions only by their smaller value at the minimum point. If we consider the autocorrelation functions of air density with initial correlation levels above 9 km, we can see that they are of the same pattern in both half-years and latitudinal zones. Therefore, all these functions can be classed in three groups. The first group includes tropospheric autocorrelation functions of air density for the middle latitudes, the second -- tropospheric autocorrelation functions of air density for the high latitudes, and the third -- all these stratospheric autocorrela- /79
tion functions of air density in both latitudinal zones and both half-years.

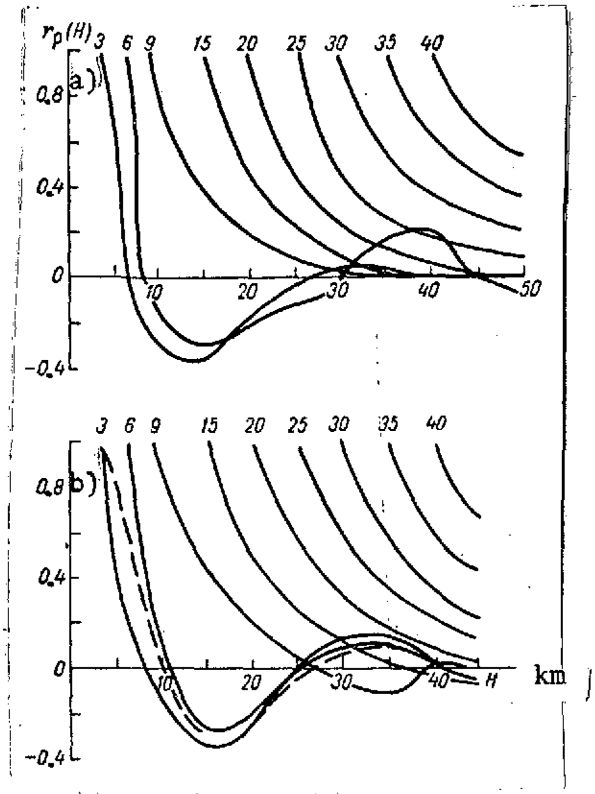


Fig. 3.19. Correlation functions of air density for the cold (a) and warm (b) half-years of the high latitudes

Now let us turn to the reciprocal correlation matrices of temperature, pressure, and air density. The reciprocal correlation matrices are conveniently represented as fields of isocorrelation. These fields make it possible to graphically represent the regions with different degrees of correlation between these atmospheric parameters within the limits of the atmospheric layers under study.

Reciprocal correlation matrices of air pressure and temperature for the cold and warm half-years in the middle latitudes are shown in Figs. 3.20 and 3.21. They have the following features:

1. The air pressure in the upper part of the troposphere as a close positive correlation with temperature at the lower-lying levels.

2. A relatively narrow /80
atmospheric layer is observed near the 15 km level in which the temperature has a fairly negative correlation with the pressure at the lower-lying levels. In the 10-15 km atmospheric layer there are very large vertical gradients of the correlation coefficient.

3. In the stratosphere the air pressure at the levels 40-50 km exhibit a positive correlation with air temperature at the levels 25-35 km.

In the high latitudes the reciprocal correlation matrices of air pressure and temperature (Figs. 3.22 and 3.23) differ somewhat from the analogous matrices for the middle latitudes. These differences pertain mainly to the stratosphere and amount to the following: 1) at the boundary between the stratosphere and troposphere there is no region with a large negative correlation between pressure and temperature; 2) in the cold half-year the region with high positive correlation between pressure in the upper stratosphere and temperature in the 30-40 km atmospheric layer is observed; 3) in the high latitudes the correlation between pressure in the /82
stratosphere and temperature at lower levels is even closer.

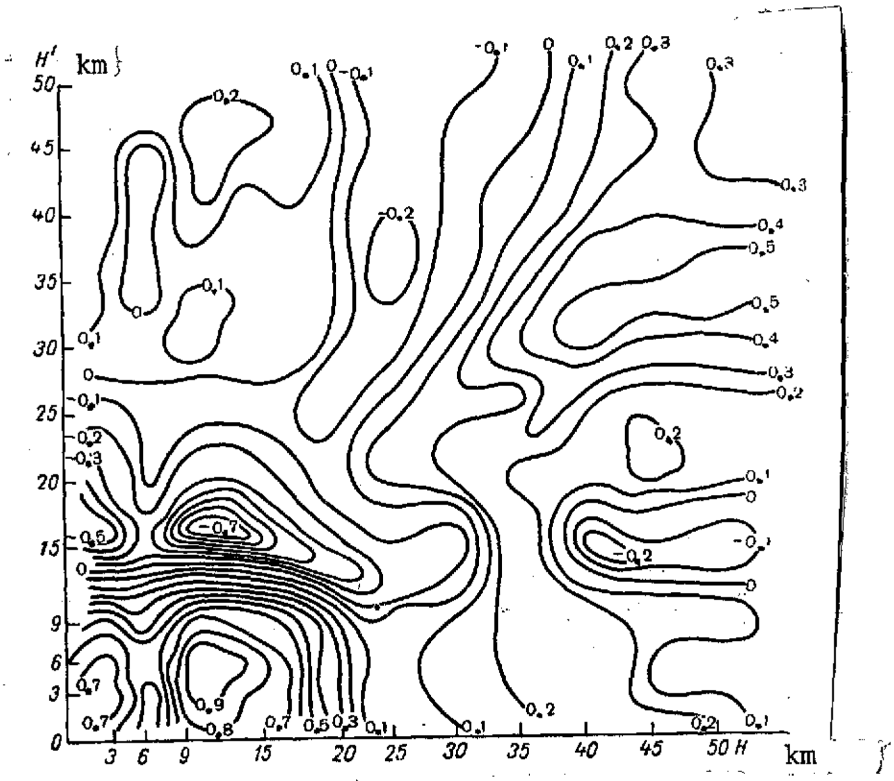


Fig. 3.20. Field of isocorrelations $r_{pt}(H, H')$ in the mid-latitude cold half-year

Figs. 3.20-3.23 clearly show also the values of the elements lying along the main diagonal of the reciprocal correlation matrices of pressure and temperature characterizing the statistical relationship between the physical parameters of the atmosphere at the same levels. Also, from these figures it follows that the 10 km correlation coefficient between temperature and pressure is 0.6-0.7. With increase in altitude, the correlation between them falls off and in the stratosphere changes little, where in the middle latitudes it is virtually absent, while in the high latitudes if one considers the bursts at the altitude 35 km in the warm half-year, it is small. In the middle latitudes there is a high negative correlation in both half-year at the altitude 15 km.

Reciprocal correlation matrices of air density and temperature have a number of interesting features.

Figs. 3.24 and 3.25 show the reciprocal correlation matrices of air density and temperature for the warm and cold half-year in the middle latitudes. From these figures it follows that the warm and cold half-years have an analogous structure. In the

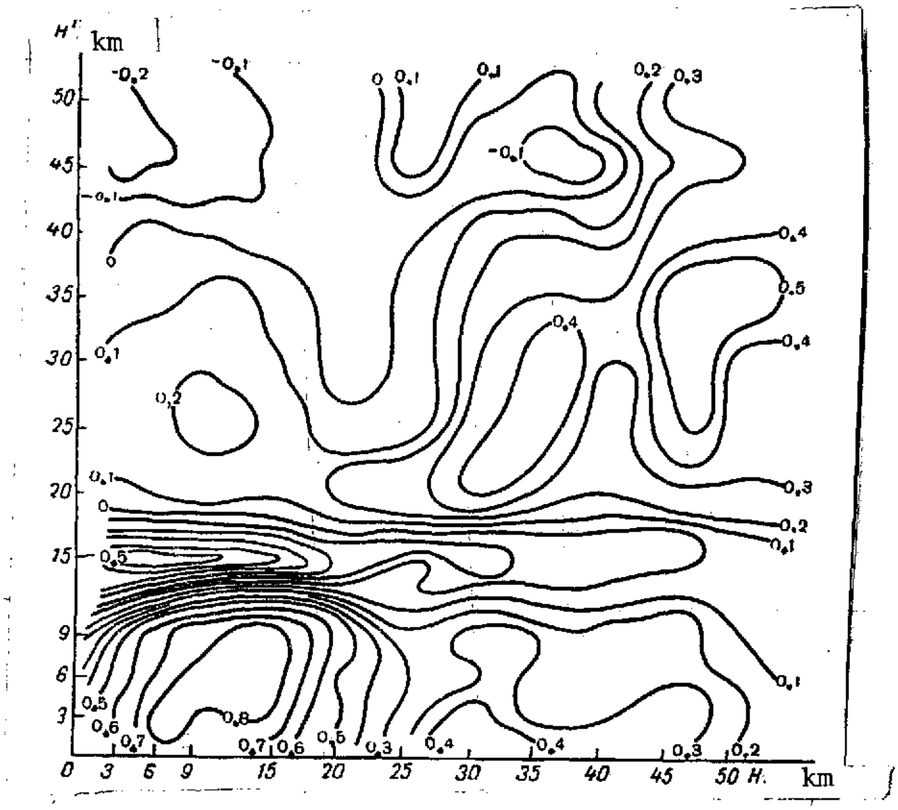


Fig. 3.21. Field of isocorrelations $r_{pt}(H,H')$ in the mid-latitude warm half-year

troposphere we observe a high reciprocal relationship between air temperature and density. A second locus of the reciprocal correlation occupies the atmospheric layer from 10 to 20 km and extends along the principal diagonal. In the isocorrelation field we observe a locus of a very high positive correlation. It indicates that an increase in air density in the troposphere corresponds to a rise in temperature at altitudes close to 15 km. In the stratosphere above 20 km the correlation between air density and temperature is very low.

The isocorrelation fields for the high latitudes (Figs. 3.26 and 3.27) differ widely from the above-considered fields. This difference lies in the fact that first of all the regions of positive correlation between density in troposphere and temperature at its boundary with the stratosphere are less intense, but have greater extent and consist of two loci, one of which is at the altitudes 10-12 km, and the second -- at the altitudes 17-22 km. Secondly, the air density in the stratosphere in the 20-40 km layer has a fairly high correlation with air temperature at the altitudes 35-40 km.

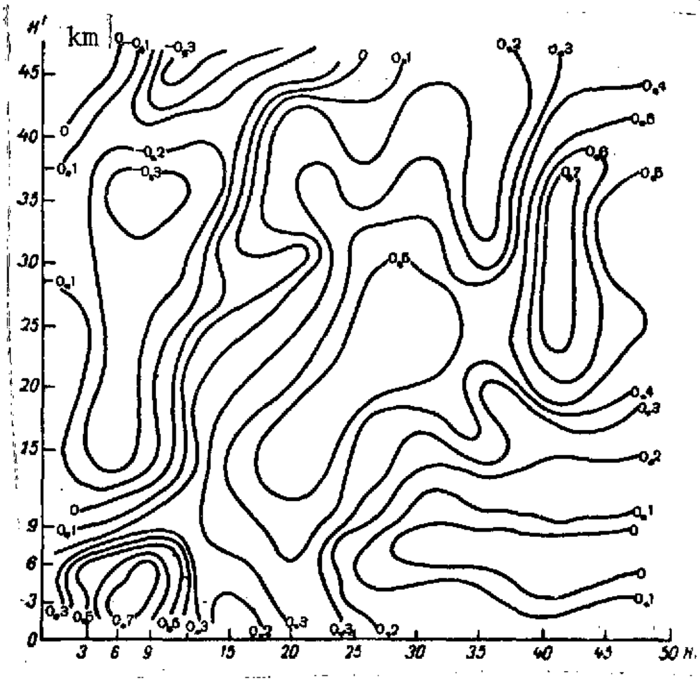


Fig. 3.22. Field of isocorrelations $r_{pt}(H, H')$ in the cold half-year of the high latitudes

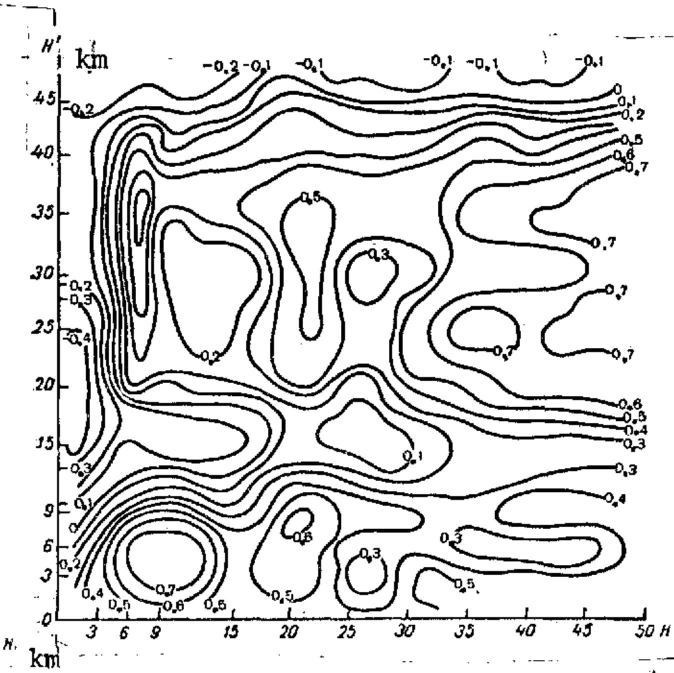
 /81

Fig. 3.23. Field of isocorrelations $r_{pt}(H, H')$ in the warm half-year of the high latitudes

The values of the diagonal elements of the reciprocal correlation matrices of air density and temperature characterizing the extent of these statistical relationship between these physical parameters of the atmosphere at the same levels indicate that between the air density and temperature at fixed levels there is a high reciprocal relationship only in the middle latitudes up to the altitude 20 km, while in the high latitudes this is true only in the lower and middle troposphere. Higher up, the correlation between them virtually disappears.

Now let us turn to the reciprocal correlation matrices of air density and pressure. They are shown in Figs. 3.28-3.31 in the form of fields of isocorrelations for the warm and cold half-years of the middle and high latitudes. All the matrices presented in these figures have the same properties, which are that in the troposphere the correlation between air density and pressure is low. In addition, a small locus of negative correlation coefficients occupies the lower left section of the isocorrelation field and indicates that a decrease in air density in the upper troposphere and in the lower part of the stratosphere usually accompanies a pressure rise in the lower troposphere. Above the tropopause is located an elongated zone of very high correlation /87

/83

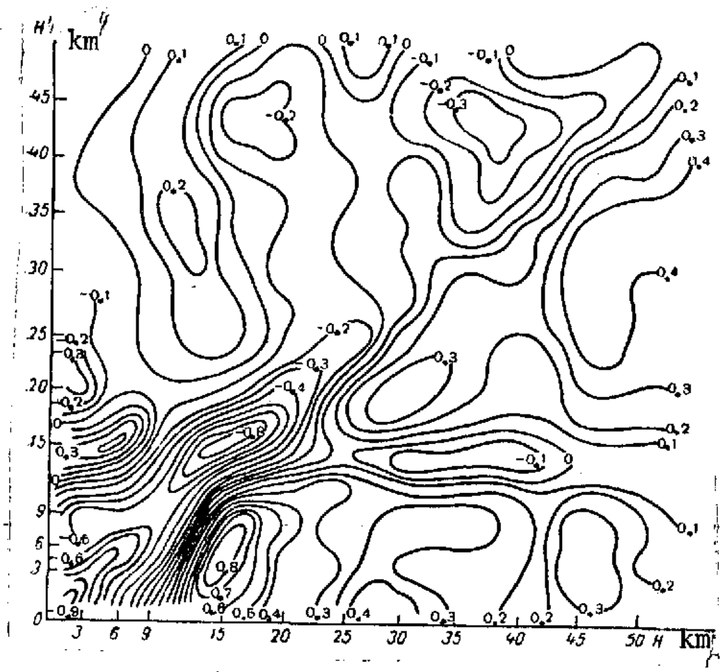


Fig. 3.24. Field of isocorrelations $r_{pt}(H, H')$ in the mid-latitude warm half-year

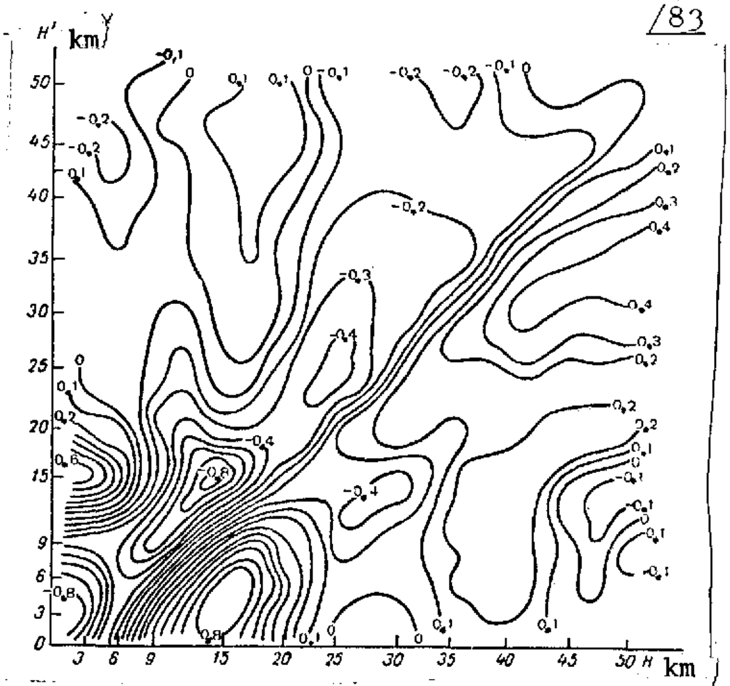


Fig. 3.25. Field of isocorrelations $r_{pt}(H, H')$ in the mid-latitude cold half-year

/84

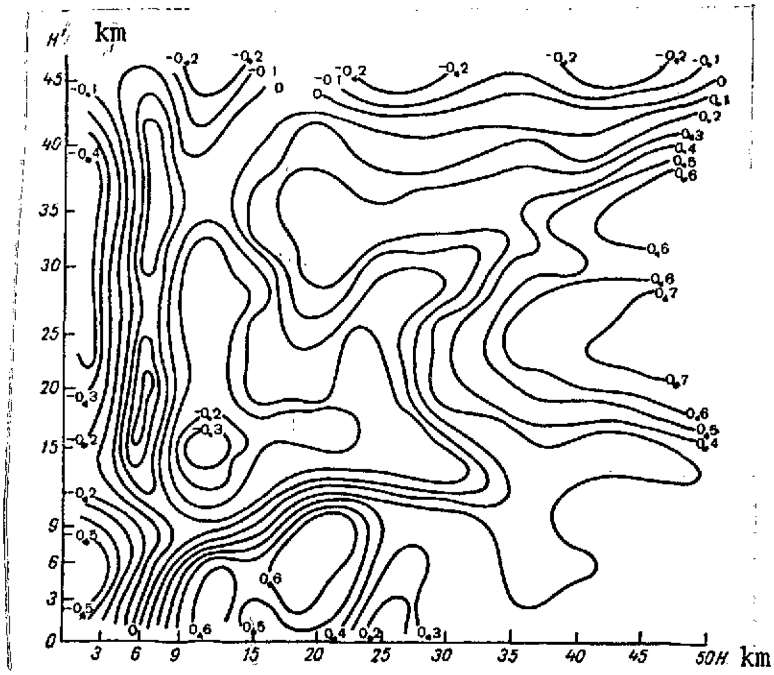


Fig. 3.26. Field of isocorrelations $r_{pt}(H, H')$ in the high-latitude warm half-year

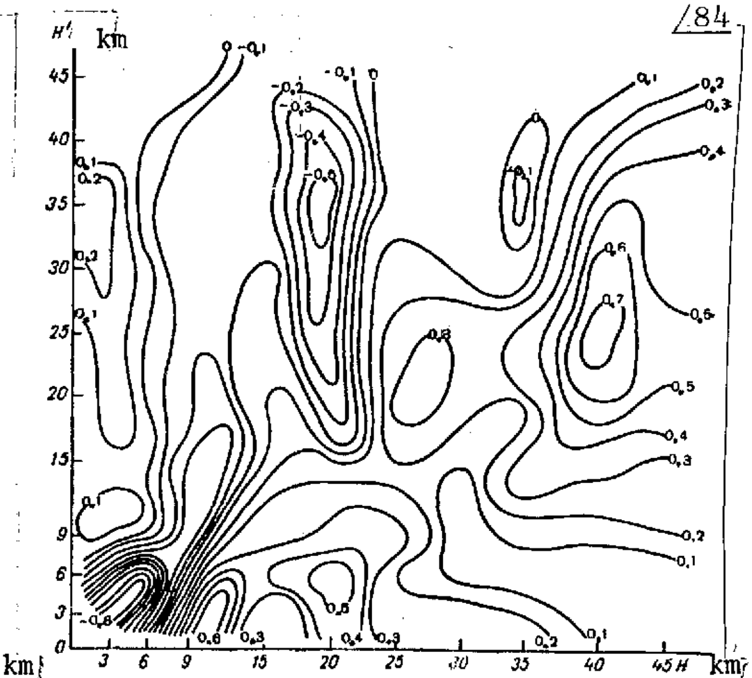


Fig. 3.27. Field of isocorrelations $r_{pt}(H, H')$ in the high-latitude cold half-year

/85

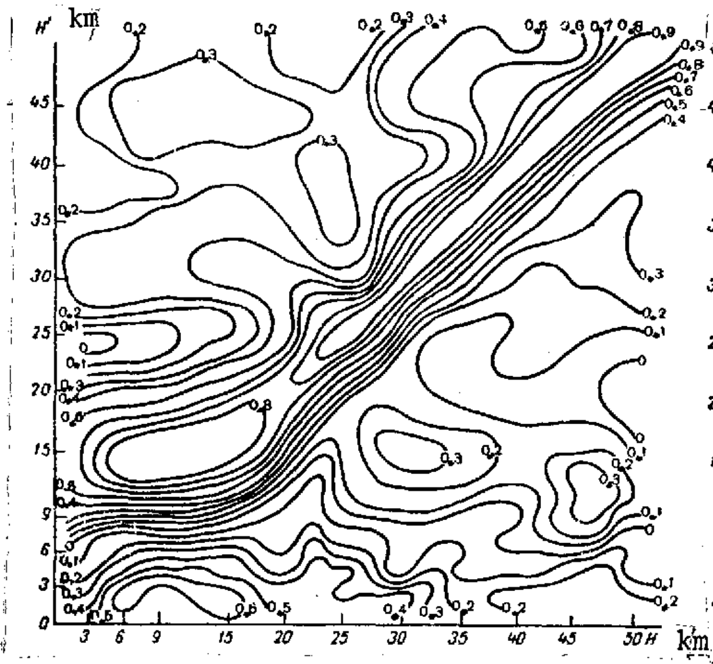


Fig. 3.28. Field of isocorrelations $r_{pt}(H, H')$ in the mid-latitude warm half-year

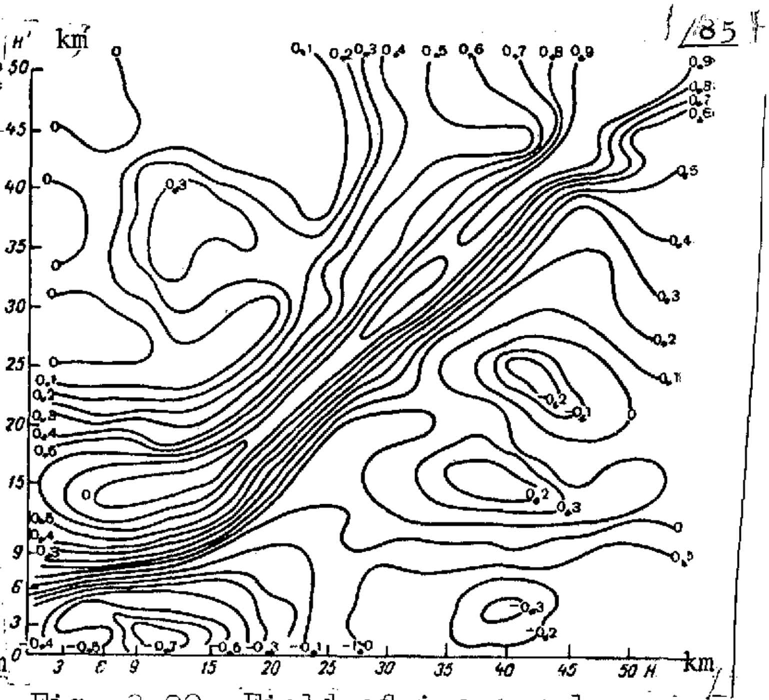


Fig. 3.29. Field of isocorrelations $r_{pt}(H, H')$ in the mid-latitude cold half-year

/86

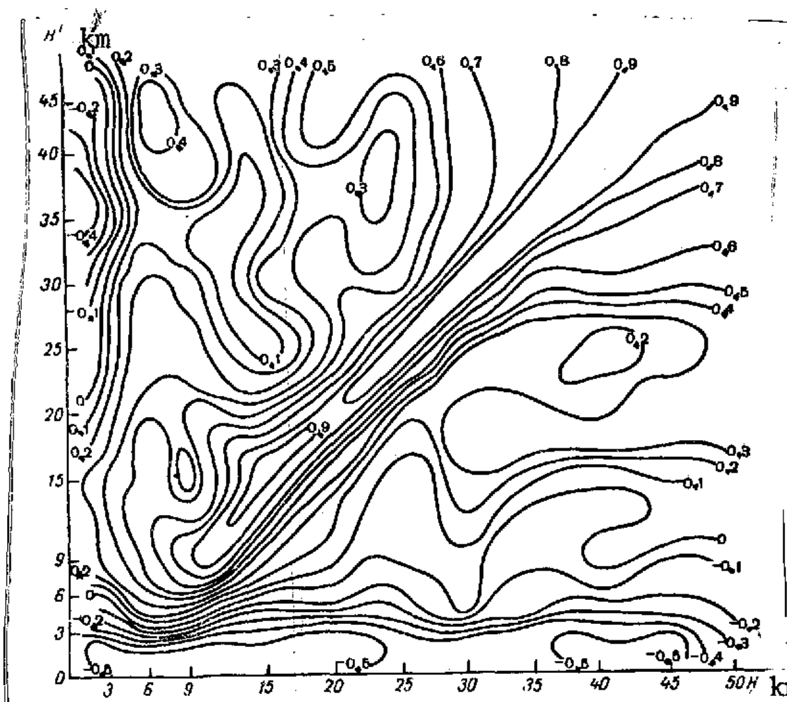


Fig. 3.30. Field of isocorrelations $r_{\rho p}(H, H')$ in the high-latitude warm half-year

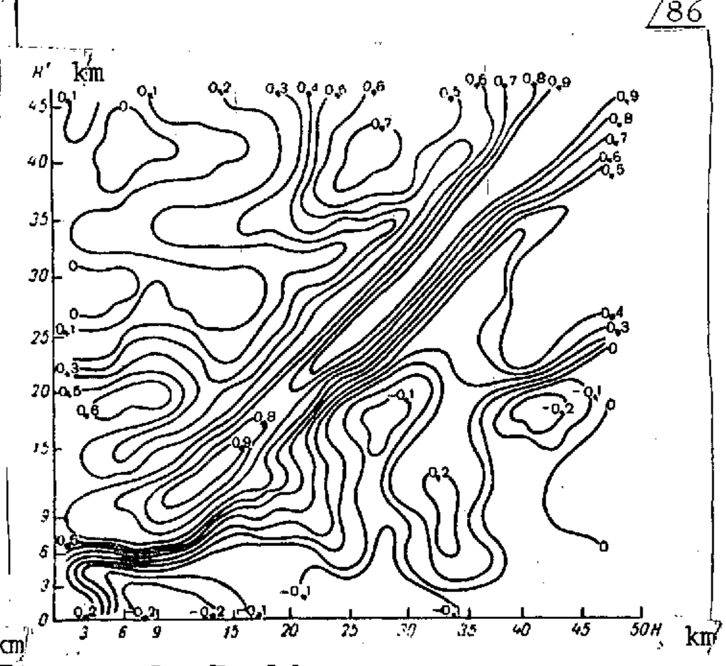


Fig. 3.31. Field of isocorrelations $r_{\rho p}(H, H')$ in the high-latitude cold half-year

coefficients between air density and pressure 10 km and more in thickness, where the maximum values lie along the principal diagonals of these matrices.

The above-noted features of the mutual correlation matrices reflect the physical relationships that are characteristic of the atmosphere. Several of these features, as for example the high correlation coefficients between pressure and temperature at the lower-lying levels and others are predictable, while the rest thus far have not found any convincing explanation.

3.4. Statistical Characteristics of Wind in Dense Atmospheric Layers

Let us examine as statistical characteristics describing the vertical structure of the wind field in dense atmospheric layers the mean values of the meridional and zonal components of wind velocity, the root mean square deviations, and the ortho- and reciprocal correlation matrices. The above-listed statistical characteristics were obtained on the basis of atmospheric rocket sounding data, briefly characterized in Section 3.1.

For these two half-years, the altitude distribution of the mean values of the zonal v and meridional u wind velocity components is shown in Figs. 3.32 and 3.33.

In the middle altitudes (Fig. 3.32), in the cold half-year the zonal component is positive in the entire atmospheric layer under consideration, which is accounted for by the prevalence in the lower 70 km atmospheric layer of a westerly transport, caused as indicated above by the circumpolar cyclonic eddy. In the troposphere the westerly component rises with altitude, reaching its maximum at the altitude 10 km. Above 10 km the westerly component diminishes and the main value occurs at the altitude 20 km. Above 20 km the westerly component began rises and becomes 68 m/sec at altitudes 70 km.

The meridional component in the entire atmospheric layer to 70 km is positive and has a relatively small value. The largest value of the meridional component occurs at the altitude 50 km and reaches 11.3 m/sec. Thus, in the middle latitudes the westerly transport with a small southerly component prevails in the cold half-year. /88

In the warm half-year the meridional component in the middle latitudes is also positive, but has an even smaller value.

The zonal component of the wind in the warm half-year rises with altitude in the troposphere. Its maximum, 11 m/sec, occurs, as in the cold half-year, at the altitude 10 km. Above this level, the westerly component diminishes and becomes equal to zero at an altitude of about 18 km. A further increase in altitude is associated with the reversal and growth of the zonal component. It reaches its maximum value, 32 m/sec, at the altitude 70 km. Therefore, in most of the stratosphere in the warm half-year quite intense easterly transport of air masses is observed.

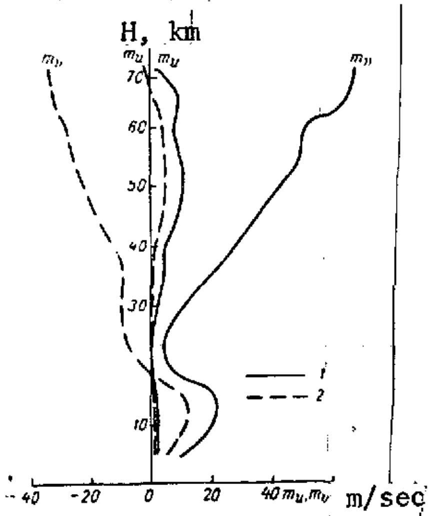


Fig. 3.32. Mean values of the wind velocity components. Middle latitudes
1. cold period
2. warm period

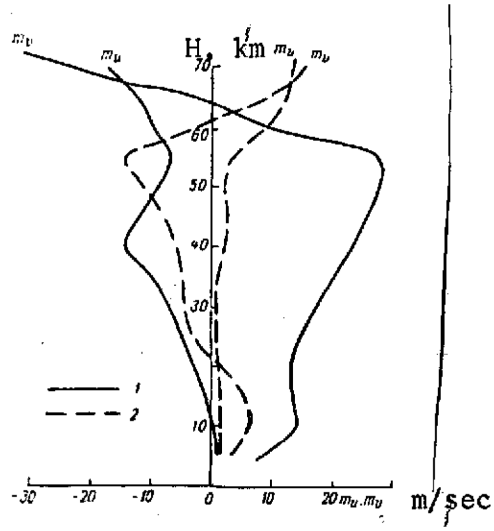


Fig. 3.33. Mean values of the wind velocity components. High latitudes
1. cold period
2. warm period

In the high latitudes (Fig. 3.33) the zonal component in the cold half-year has the following features. Just as in the middle latitudes, it is positive up to the altitude 10 km and rises with altitude, while it remains nearly unchanged in the 10-20 km layer. Above 20 km an increase occurs in the velocity of the westerly wind. At the altitude 55 km it reaches a maximum, 28 m/sec. A decrease in the zonal component accompanies a further increase in altitude, where at the altitude 75 km it becomes negative and rises with altitude.

The meridional wind velocity component is negative throughout nearly the entire 70-km atmospheric layer. Its greatest value, -18 m/sec, lies at the altitude 70 km. /89

In the warm half-year the zonal component of wind velocity in the troposphere in the high latitudes is also positive up to the altitude 20 km. Above this altitude its reversal and increase is observed. The largest negative value of the zonal components (-15 m/sec) occurred at the altitude 55 km. Upwards of this, it again becomes positive and rises.

The meridional component is positive and small in the entire atmospheric layer under consideration. Only above 50 km does it rise appreciably, reaching 50 m/sec at the altitude 70 km. The values of the wind velocity components indicate that in the warm half-year in the high latitudes easterly winds also prevail, which in the lower mesosphere change into westerly winds.

By comparing Figs. 3.32 and 3.33 with the time profiles of the wind field for 30 and 60° N. Lat (see Figs. 2.1 and 2.2), we can note that the altitude distributions of the wind velocity components averaged by half-years agrees fully with the time profile. The somewhat smaller maxima of wind velocity components are associated with the additional smoothing occurring during averaging by half-years and by latitudes.

Fig. 3.34 shows the variation with altitude of the root mean square deviations of wind velocity in the middle and high latitudes. The graph showed that the distribution of the root mean square deviations of the wind velocity components differs in the different zones. In the middle latitudes the root mean square deviations of the wind velocity components are nearly identical up to the altitudes 20-25 km in the cold and warm half-years. Above these levels, the root mean square deviations of the zonal component become larger than the meridional, by a factor of 2-3. Here the root mean square deviations of the wind velocity components in the cold half-year exceed their values in the warm half-year. The largest value of the root mean square deviations of the meridional component (22 m/sec) is observed in the cold half-year at the altitude 75 km, and the zonal component (33 m/sec) -- at the altitude 60 km.

In the high latitudes the altitude distribution of the root mean square deviations of wind velocity components in the cold half-year differs appreciably from their distribution in the warm half-year. While in the warm half-year the ratios between the root mean square deviations of zonal and meridional components are nearly the same as in the middle altitudes, in the cold half-year the root mean square deviations of both components are nearly identical in the entire atmospheric layer under study.

Generalized covariance and correlation matrices of the wind velocity, as shown above, consist of four blocks. Blocks located along the principal diagonals of the generalized correlation matrix are autocorrelation matrices of the zonal and meridional wind velocity components. For purposes of analyzing the matrices, let us represent also in the form of autocorrelation functions. Let us look at some features of these functions. /90

Fig. 3.35 shows the autocorrelation functions of the zonal $r_v(\Delta H)$ and meridional $r_u(\Delta H)$ wind velocity components in the mid-latitude cold half-year. From Fig. 3.35 it follows that the functions $r_u(\Delta H)$ with 5-20 km initial correlation levels, and the functions are settled with 5-25 km initial correlation level (the altitudes of the initial correlation levels in kilometers are given alongside each curve) are of identical form. Above 25 km the correlation of the zonal components falls off more slowly with altitude. /91

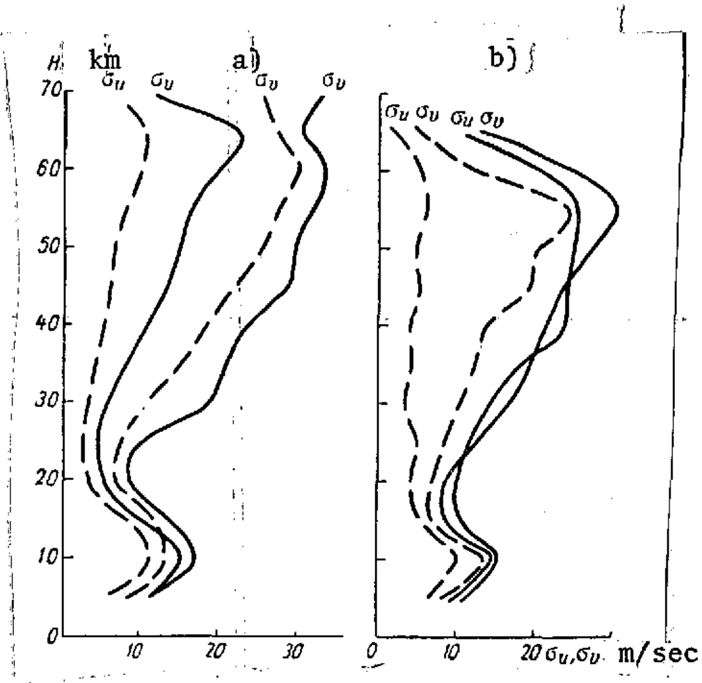


Fig. 3.34. Root mean square deviations of wind velocity components
a. middle latitudes
b. northern latitudes

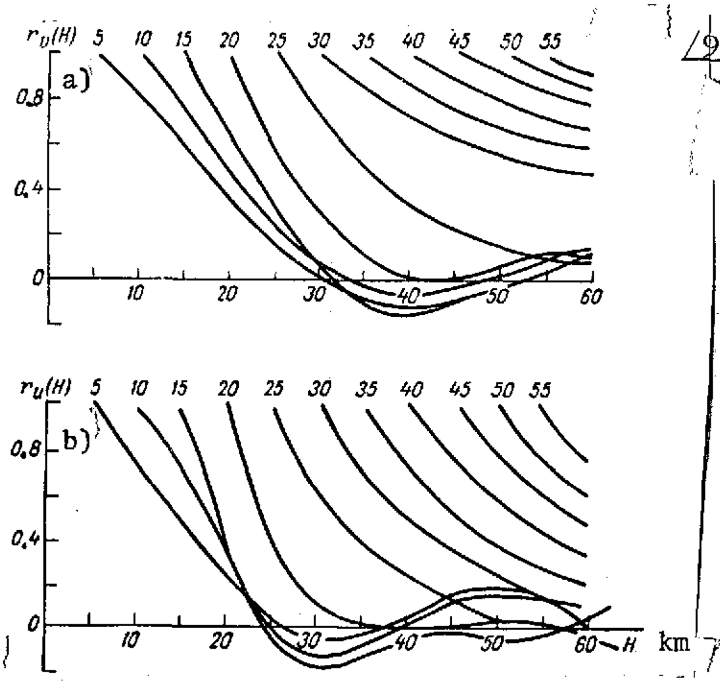


Fig. 3.35. Correlation functions of zonal (a) and meridional (b) wind velocity components in the mid-latitude cold half-year

Autocorrelation functions of wind velocity components for the warm half-year are shown in Fig. 3.36 for the middle latitudes; from this figure it is clear that in the warm half-year the pattern of the variation in correlation with altitude shown by meridional and zonal components is different. If the functions for the meridional components with 5-15 km initial correlational levels are of the same form as in the cold half-year for the same altitudes, functions with initial correlation levels higher than 15 km indicate a rapid falloff in the correlation of the meridional components /92 with increase in altitude. Conversely, autocorrelation functions of these zonal component decrease very slowly.

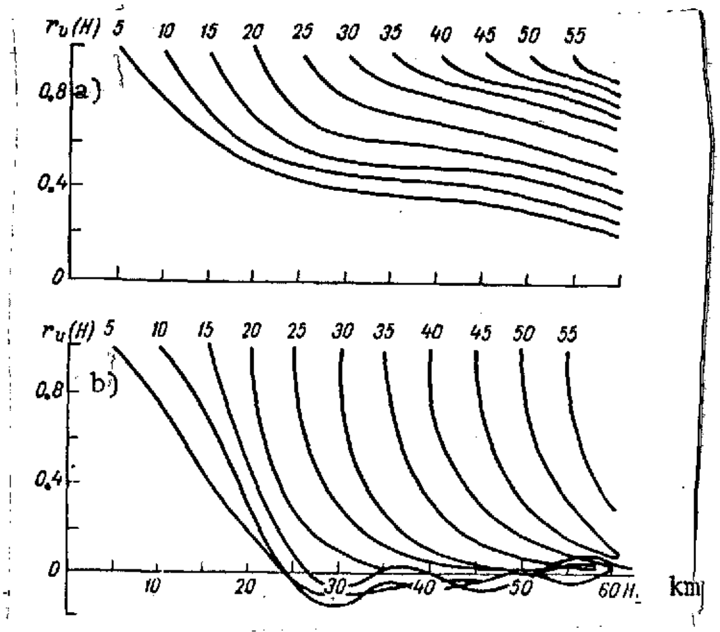


Fig. 3.36. Correlation functions of zonal (a) and meridional (b) wind velocity components in the high-latitude warm half-year

In the cold half-year the autocorrelation of the zonal (Fig. 3.37 a) and meridional (Fig. 3.37 b) wind velocity components in the high latitudes are virtually identical in form throughout this entire atmospheric layer.

In the warm half-year the autocorrelation functions of wind velocity components in the high latitudes are virtually analogous to the corresponding functions in the middle latitudes in the warm half-year. The correlation between the meridional components (Fig. 3.38 b) with increase in altitude above 15 km falls off very rapidly, but between these zonal components (Fig. 3.38 a) -- very slowly in this entire atmospheric layer.

If we consider jointly all autocorrelation functions of wind velocity components, we can note that their entire set can be subdivided into four homogeneous groups. The first group includes the autocorrelation functions of the meridional wind velocity component for the warm and cold half-years in the middle latitudes and for the warm half-year in the high latitudes, and the zonal component for the mid-latitude cold half-year, which has 5-20 km altitudes for their initial correlation levels. The second group contains the autocorrelation functions of the meridional wind velocity component with initial correlation levels higher than 20 km for the middle and high latitude warm half-year. The third group unites all the autocorrelation functions of the meridional and zonal components in the mid-latitude cold half-year with initial correlation levels higher than 20 km. Finally, the fourth group includes the autocorrelation functions of the zonal wind velocity component for the mid- and high-latitude warm half-year, and also for the mid-latitude cold half-year above 20 km. /93

The reciprocal correlation matrices of the wind velocity components for both half-years and both latitudinal groups are identical in pattern. This pattern amounts to the virtual absence of a correlation between the wind velocity components either at the same levels or at different levels, within this particular atmospheric layer (Table 3.9).

TABLE 3.9. RECIPROCAL CORRELATION MATRIX OF WIND VELOCITY COMPONENTS $r_{vu}(H, H')$. WARM HALF-YEAR, MIDDLE LATITUDES /94

H km	5	10	15	20	25	30
5	0.21	0.20	0.10	0.09	0.04	0.02
10	0.22	0.25	0.13	0.02	0.03	–0.02
15	0.20	0.24	0.16	0.03	0.02	0.02
20	0.07	0.09	0.02	0.00	–0.11	–0.06
25	0.01	0.08	0.07	0.01	0.04	0.00
30	–0.12	–0.07	–0.03	–0.06	–0.06	–0.06
35	–0.07	–0.03	0.01	0.04	–0.05	–0.05
40	–0.07	–0.09	–0.04	–0.02	–0.04	0.02
45	0.11	0.14	0.08	0.10	0.12	0.12
50	–0.04	–0.03	–0.07	–0.01	0.01	0.03
55	–0.06	–0.08	–0.09	–0.05	–0.04	–0.07
60	0.07	0.08	0.06	0.04	0.05	0.08

H km	35	40	45	50	55	60
5	0.05	0.06	0.06	0.05	0.01	–0.03
10	0.02	0.02	0.00	0.01	–0.05	–0.04
15	0.08	0.06	0.02	0.04	–0.03	–0.06
20	–0.03	–0.06	–0.04	–0.02	–0.04	–0.03
25	0.01	–0.01	0.01	0.03	0.01	0.00
30	–0.08	–0.11	–0.05	–0.03	–0.03	–0.01
35	–0.05	–0.04	–0.02	–0.12	–0.03	–0.01
40	–0.01	–0.01	0.01	0.02	0.08	0.03
45	0.18	0.12	0.16	0.18	0.16	0.05
50	0.08	0.07	–0.01	0.08	0.09	0.01
55	–0.01	–0.04	–0.01	–0.07	–0.02	0.01
60	0.10	0.12	0.15	0.11	0.13	0.14

The above-described features of the autocorrelation matrices, of the altitude distributions of the root mean square deviations, and the nature of the correlations between the wind velocity components lead to the conclusion that the scatter of wind velocities can be regarded as circular in the middle latitudes and in the warm half-year in the high latitudes approximately to the altitudes 20-25 km, as well as in this entire atmospheric layer during the /95 cold half-year in the high latitudes. In the remaining cases, the scatter of wind velocities has a well-defined ellipticity. Here, since the correlation between the wind velocity components is virtually absent, the axes of the scatter ellipse coincide with the axes of the standard coordinate system, which as indicated above, was used for expanding the wind velocity into components.

3.5. Precision of Determining the Statistical Characteristics of the Physical Parameters of the Atmosphere

The values of the atmospheric parameters measured during the sounding periods with meteorological rockets contain errors of two kinds. The first kind includes systematic errors, for example, dynamic and inertial. They are cancelled out by introducing appropriate corrections into the measurement data. The second kind of errors -- random errors -- is contained in the initial data. As a result, the statistical characteristics calculated from these data contain certain errors. Also, since the statistical characteristics were calculated for restricted sets, as indicated above, they are estimates of the unknown functions, that is, they have a certain precision.

If we denote the random variable under study by x, then its mean $\bar{x}$ obtained for some set n will be the closer to its mathematical expectation m_x the larger the n. The difference between them can be characterized by the root mean square error of the mean $\sigma_{\bar{x}}$, which for normally distributed values of x is defined by the following expression:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}. \tag{3.22}$$

When the number of cases in the set under study is larger than several tens, the difference between the biased and unbiased estimates virtually disappears. If we denote the dispersion of the unknown random variable by $\tilde{D}_x$, then

$$m[\tilde{D}_x] = D_x; \quad D[\tilde{D}_x] = \frac{2}{n} D_x^2. \tag{3.23}$$

The right expression characterizes the precision of determining the dispersion of variable x. Similarly, for the covariance of random variables x and y we have

$$m[\tilde{R}_{xy}] = R_{xy}; \quad D[\tilde{R}_{xy}] = \frac{D_x D_y + R_{xy}^2}{n}. \tag{3.24}$$

The second formula of equalities (3.24) allows us to obtain the root mean square error of the covariance /96

$$\sigma[\tilde{R}_{xy}] = \frac{\sigma_x \sigma_y}{\sqrt{n}} \sqrt{1 + r_{xy}^2}. \qquad (3.25)$$

We can also compute the precision with which the root mean square deviation of the random variable x was calculated. To do this, in the case of the normal distribution we have the formula

$$\sigma_{\sigma_x} = \frac{\sigma_x}{\sqrt{2n}}. \qquad (3.26)$$

The degree of approximation of the estimates to the unknown statistical characteristics is customarily estimated by using confidence intervals. To determine the confidence intervals of the computed statistical characteristic, we must assign the probability with which the given characteristic will fall in these intervals. If we assume, for example, that this probability is 0.95 (this probably corresponds to the familiar rule 2), the probability of the opposite event is 0.05.

Let us consider the variable

$$k = \frac{\overline{X} - m_X}{\sigma_{\overline{X}}}.$$

Then we can write that the probability is

$$P\{|k| \geqslant k_{0,05}\} = 1 - \Phi(k_{0,05}) = 0{,}05, \qquad (3.27)$$

where $\Phi(k_{0,05})$ is the probability integral.

The quantity $k_{0.05}$ can be found from appropriate tables and is 1.96 for the specified probability. Then Eq. (3.27) can be written as follows:

$$P(|\overline{X} - m_X| \geqslant 1{,}96\sigma_{\overline{X}}) = 0{,}05. \qquad (3.28)$$

This equality allows us to state that with a probability 0.95, we have

$$\overline{X} - 1{,}96\sigma_{\overline{X}} < m_X < \overline{X} + 1{,}96\sigma_{\overline{X}}. \qquad (3.29)$$

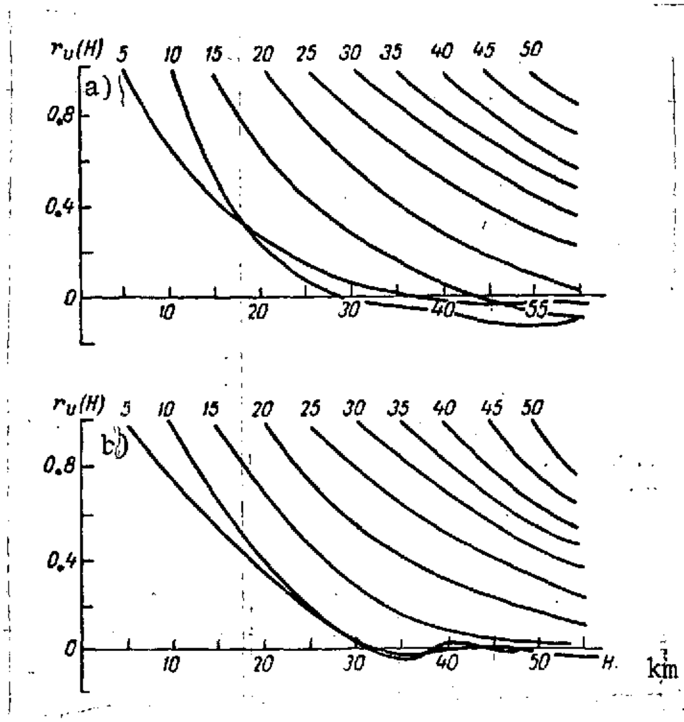


Fig. 3.37. Correlation functions of zonal (a) and meridional (b) wind velocity components in the high-latitude cold half-year

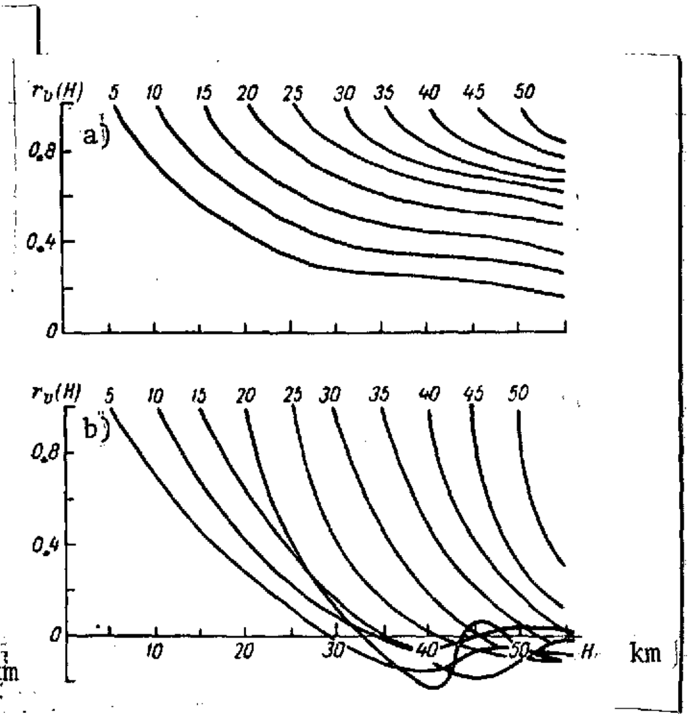


Fig. 3.38. Correlation functions of zonal (a) and meridional (b) wind velocity components in the high-latitude warm half-year

TABLE 3.10. CONFIDENCE INTERVALS OF AIR DENSITY (g/m3), TEMPERATURE (°C), AND PRESSURE (mb) FOR THE WARM-YEAR OF THE NORTH AMERICAN CONTINENT /97

H, km	Middle latitudes						High latitudes					
	$2\sigma_{\sigma_\rho}$	$2\sigma_{\bar{\rho}}$	$2\sigma_{\sigma_t}$	$2\sigma_{\bar{t}}$	$2\sigma_{\sigma_p}$	$2\sigma_{\bar{p}}$	$2\sigma_{\sigma_\rho}$	$2\sigma_{\bar{\rho}}$	$2\sigma_{\sigma_t}$	$2\sigma_{\bar{t}}$	$2\sigma_{\sigma_p}$	$2\sigma_{\bar{p}}$
3	1.120	1.640	0.46	0.64	0,392	0.560	3.341	4.870	0.92	1.36	2.615	3.750
6	0.710	1.000	0.46	0.64	0,520	0.740	2.251	3.236	1.10	1.56	1.894	2.694
9	0.700	0.994	0.56	0.80	0,561	0.802	2.052	3.122	0.78	1.12	1.700	2.416
15	0.912	1.298	0.52	0.76	0,310	0.444	1.500	2.130	0.62	0.92	0.885	1.264
20	0.231	0.335	0.46	0.66	0,135	0.192	0.374	0.528	0.62	0.42	0.258	0.368
25	0.397	0.422	0.47	0.67	0,193	0.274	0.412	0.588	0.83	1.18	0.284	0.406
30	0.108	0.154	0.56	0.82	0,073	0.106	0.396	0.566	1.48	2.12	0.280	0.400
35	0.058	0.082	0.70	1.00	0,042	0.060	0.109	0.155	1.29	1.84	0.090	0.128
40	0.059	0.084	1.10	1.56	0,043	0.062	0.076	0.108	1.29	1.84	0.056	0.086
45	0.021	0.030	0.83	1.26	0,016	0.023	0.045	0.065	1.33	1.94	0.035	0.050
50	0.011	0.016	0.98	1.40	0,009	0.013	0.034	0.048	1.43	2.04	0.025	0.036
55	0.014	0.020	0.82	1.18	0,013	0.019	0.043	0.061	1.34	1.92	0.033	0.047
60	0.004	0.006	1.10	1.54	0,007	0.010	0.034	0.049	0.81	1.16	0.028	0.040
65	0.004	0.006	2.10	3.00	0,007	0.009	0.008	0.011	0.20	0.28	0.008	0.012

Similar arguments can be presented also for determining the confidence intervals of the root mean square deviations.

Table 3.10 presents the confidence intervals of the means and root mean square deviations of air pressure, temperature, and density that were considered above in order to describe the features of the vertical statistical structure of fields of these physical parameters of the atmosphere.

The confidence intervals for the wind velocity components are found in Table 3.11.

TABLE 3.11. CONFIDENCE INTERVALS OF WIND VELOCITY COMPONENTS (m/sec). NORTH AMERICA, WARM HALF-YEAR

H, km	Middle latitudes				High latitudes			
	$2\sigma_{\sigma_u}$	$2\sigma_{\bar{u}}$	$2\sigma_{\sigma_v}$	$2\sigma_{\bar{v}}$	$2\sigma_{\sigma_u}$	$2\sigma_{\bar{u}}$	$2\sigma_{\sigma_v}$	$2\sigma_{\bar{v}}$
5	0,44	0,62	0,57	0,82	1,00	1,42	1,33	1,90
10	0,79	1,12	0,90	1,28	1,58	2,24	2,16	3,08
15	0,63	0,90	0,80	1,14	0,82	1,16	0,98	1,40
20	0,22	0,30	0,43	0,60	0,60	0,86	1,08	1,54
25	0,20	0,28	0,53	0,76	0,76	1,08	1,33	1,90
30	0,26	0,38	0,77	1,10	0,56	0,80	1,64	2,34
35	0,32	0,46	1,07	1,52	0,63	0,90	1,87	2,66
40	0,36	0,42	1,20	1,70	0,62	0,88	2,08	2,98
45	0,43	0,66	1,50	2,12	0,78	1,12	2,84	4,20
50	0,34	0,48	1,76	2,20	0,74	1,06	2,84	4,20
55	0,56	0,70	1,92	2,74	0,98	1,40	3,73	5,32
60	0,76	1,08	2,11	3,02	0,98	1,40	1,80	2,68
65	1,10	1,42	1,18	1,68	0,45	0,64	0,94	1,34
70	3,81	5,81	2,30	3,30				

From Table 3.11 it follows that these statistical characteris- /98
tics exhibit a precision that is satisfactory for practical purposes. The confidence intervals for the cold half-year differ little from those presented above and therefore are not given here.

The root mean square error of the covariant matrix elements r_{xy} is determined by using the formula

$$\sigma_r = \frac{1 - r_{xy}^2}{\sqrt{n}}, \tag{3.30}$$

Obtained under the assumption that the law of the distribution of variables is normal. An investigation by Fisher /31/ shows that the probability density of the correlation coefficient r has the following form:

$$f_n(r) = \frac{n-2}{\pi}(1-\rho^2)^{\frac{n-1}{2}}(1-r^2)^{\frac{n-1}{2}}\int_0^1 \frac{\xi^{n-2}}{(1-\rho r\xi)^{n-1}}\frac{d\xi}{\sqrt{1-\xi^2}}. \tag{3.31}$$

From Eq. (3.31) it follows that the distribution of the correlation coefficients is not normal. It depends not only on the size of the sample n, but also on the correlation coefficient of the general set rule. The distribution (3.31) approaches the normal distribution only for small r and for the large n. Under these conditions, Eq. (3.30) is also applicable.

To determine the confidence intervals of the correlation coefficient, Fischer proposed the transformation

$$z = \frac{1}{2}\ln\frac{1+r}{1-r}. \tag{3.32}$$

The value of z even for small n is distributed normally with amea mean and with dispersion, determined by using the approximate equalities

$$m[z] = \xi + \frac{\rho}{2(n-1)}; \quad D_z = \frac{1}{n-3}; \tag{3.33}$$

where

$$\xi = \frac{1}{2}\ln\frac{1+\rho}{1-\rho}. \tag{3.34}$$

If we find the root mean square deviation of the z-transformation, we can determine the confidence interval of z:

$$z_1 < z_0 < z_2; \tag{3.35}$$

where z_0 is the value of z for large n, and

$$z_1 = z - k\sigma_z,$$
$$z_2 = z + k\sigma_z.$$

By determining z_1 and z_2, it is not complicated by means of transformation (3.32) to compute the values of r_1 and r_2 corresponding to them for a specified probability. The latter defines a precision of the computed elements of the correlation matrices.

Table 3.12 gives the confidence intervals with a probability 0.67 of the elements of the correlation matrices of air pressure, density, and temperature, and of the wind velocity components for the warm half-year in the middle and high latitudes. /99

TABLE 3.12. CONFIDENCE INTERVALS OF CORRELATION MATRICES. NORTH AMERICA, WARM HALF-YEAR

r_0	Middle latitudes				High latitudes	
	p, ρ, t		u, v		p, ρ, t, u, v	
	r_1	r_2	r_1	r_2	r_1	r_2
0.1	0.18	0.02	0.15	0.05	0.21	−0.01
0.2	0.28	0.12	0.25	0.15	0.30	0.10
0.3	0.37	0.23	0.35	0.25	0.40	0.20
0.4	0.47	0.33	0.44	0.36	0.49	0.31
0.5	0.56	0.44	0.54	0.46	0.58	0.42
0.6	0.65	0.55	0.63	0.57	0.67	0.53
0.7	0.74	0.66	0.72	0.67	0.75	0.64
0.8	0.83	0.77	0.82	0.78	0.84	0.76
0.9	0.91	0.89	0.91	0.89	0.92	0.88

From Table 3.12 it follows that the highest error is characteristic of the smallest values of the correlation matrix elements. As their values approach unity, the precision rises appreciably.

The precision of computing a number of statistical characteristics depends also on the random errors of measurement contained in the initial data. From error theory we know that random errors do not affect to the value of the elements in the reciprocal correlation matrices, since the errors of the data dealing with different elements, as well as errors at different points do not correlate together. Moreover, the random errors exceed the dispersion elements of the **autocovariance matrices by the value of the root** mean square of the error. When interpreting the results of rocket sounding of the atmosphere, we must also take into account that for the same sounding the random errors of measurement, for example, of temperature, correlate with the measured value of the element owing to the accumulation of errors, one reason for which is the inertia of the transducers. As a result, the values of the correlation functions of air temperature obtained in interpreting the initial material proved to be overstated. The most widely accepted method of cancelling out random errors is extrapolation to zero of single-level structure functions. The available initial material does not always enable us to obtain these functions and therefore to use this technique. However, the random error can be determined if we use the properties of a simple Markov random process. Their applicability to the atmosphere was shown by M. I. Yudin /84/.

The following equalities are valid for the simple Markov /100
process:

$$\left.\begin{aligned}\tilde{R}_{m,\,m+1} &= \sigma_m \sigma_{m+1} \tilde{r}_{m,\,m+1},\\ \tilde{R}_{m,\,m+2} &= \sigma_m \sigma_{m+2} \tilde{r}_{m,\,m+1} \tilde{r}_{m+1,\,m+2},\\ \tilde{R}_{m+1,\,m+2} &= \sigma_{m+1} \sigma_{m+2} \tilde{r}_{m+1,\,m+2} \quad (m = 1, 2, \ldots).\end{aligned}\right\} \tag{3.36}$$

In equality (3.36) the corresponding statistical characteristics obtained by the statistical interpretation of the initial data are denoted with the sign ~ . Using equality (3.36), we can approximately determine the error contained in the estimates of the root mean square deviations and cancel it out. Actually, assuming that the errors in the values of the covariances are determined by errors contained in the correlation coefficients, from equality (3.36) let us determine the "true" value of the dispersion

$$\sigma^2_{m+1} = \frac{\tilde{R}_{m+1,\,m+2}\tilde{R}_{m,\,m+1}}{\tilde{R}_{m,\,m+2}} \quad (m = 1, 2, \ldots). \tag{3.37}$$

Then the dispersion of the random error ε^2_{m+1} can be computed by the formula

$$\varepsilon^2_{m+1} = \tilde{\sigma}^2_{m+1} - \sigma^2_{m+1}. \tag{3.38}$$

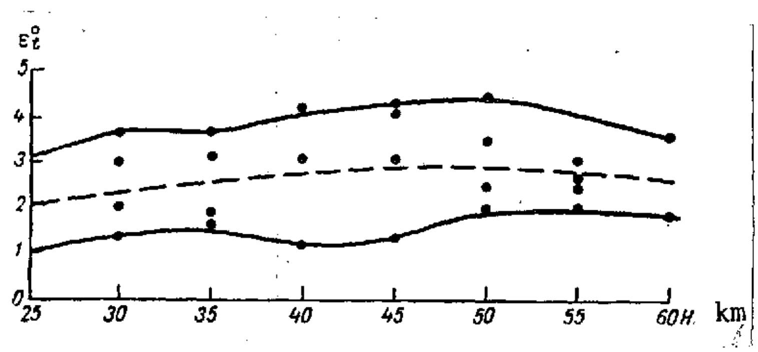


Fig. 3.39. Root mean square values of random errors of air temperature

Fig. 3.39 shows the root mean square values of random errors of temperature obtained by means of relations (3.36) - (3.38). By averaging the errors, we get the result that above 20 km the root mean squares of the random temperature errors average 2-3°C. These values agree well with the root mean square values of the random temperature errors obtained, for example, by V. P. Boltenkov, which in the layer from the ground level to 20 km were found to be 1-1.5°C. In conclusion, we know that after determining the dispersions of the random errors it is not difficult also to correct the corresponding correlation functions.

CHAPTER FOUR /101

FORMS OF REPRESENTATION OF ATMOSPHERIC PERTURBATIONS

4.1. Approximation of Autocorrelation Functions of Air Temperature and Density and of Wind With Analytic Expressions

In Chapter Three it was shown that autocorrelation functions of air temperature in specific atmospheric layers, latitudinal groups and half-years exhibit similarity, which makes it possible to combine them into four groups with given features. Autocorrelation functions of air density also can be divided into three homogeneous groups. First we consider jointly the autocorrelation functions of air temperature and density, we note a high similarity of the tropospheric and stratospheric autocorrelation functions of these meteorological elements. Therefore the autocorrelation functions of air temperature and density can be combined into four groups.

Group I includes autocorrelation functions of temperature with initial correlation levels at 3, 6, and 9 km, and air density with initial correlation levels of 3 and 6 km, related to the middle latitudes. They have the shape of a damped cosine.

Group II includes autocorrelation functions of air temperature and density for the high latitudes with initial correlation levels at 3 and 6 km. They have the same pattern as the functions in group I, but differ from the latter by their smaller amplitude.

Group III consists of autocorrelation functions of temperature for the middle latitudes with initial correlation levels at 20 km and higher in the cold half-year, and also 15 and 20 km in the warm half-year.

Group IV includes autocorrelation functions of temperature that have initial correlation levels higher than 20 km in the warm half-year in the middle latitudes, and higher than 6 km in the warm and cold half-year in the high latitudes, as well as autocorrelation functions of air density with initial correlation levels higher than 6 km in both latitudinal zones in the cold and warm half-years.

Each of these groups of homogeneous curves can be approximated with a single analytic expression. Obviously, the curve groups I and II are conveniently approximated with the expression /102

$$r(\Delta H) = e^{-k\Delta H} \cos \Omega_0 \Delta H, \tag{4.1}$$

and curve groups III and IV -- by the equality

$$r(\Delta H) = e^{-k\Delta H}. \tag{4.2}$$

Table 4.1 gives the values of k and Ω_0 obtained by the method of least squares. The different values of the coefficients correspond to the different groups of functions.

TABLE 4.1. VALUES OF k AND Ω_0 FOR AUTOCORRELATION FUNCTIONS OF AIR TEMPERATURE AND DENSITY

Coefficient	Group of functions			
	I	II	III	IV
k km^{-1}	0.040	0.094	0.157	0.0975
Ω_0 km^{-1}	0.21	0.21	—	—

TABLE 4.2. ROOT MEAN SQUARE ERRORS OF APPROXIMATIONS OF AUTOCORRELATION FUNCTIONS

Group	ΔH km				
	2	5	10	15	20
III	0.05	0.08	0.08	0.09	0.10
IV	0.04	0.07	0.11	0.11	0.12

The results of approximations of the functions in groups I and II are plotted with a dashed line in Figs. 3.15 a and 3.16 b.

Table 4.2 gives the root mean square errors of the approximations for functions in group III and IV.

The root mean square errors of the approximation have about the same value as the confidence intervals of the autocorrelation functions, which indicates good agreement between the analytic and empirical functions.

Analysis of the autocorrelation functions of wind velocity component shows that their entire set can be divided into four groups of homogeneous functions.

Group I includes autocorrelation functions of the meridional wind velocity components for the warm and cold half-years in the middle latitudes, and for the warm half-year in the high latitudes, and the zonal component of wind velocity for the cold half-year in the middle latitudes, which have initial correlation levels at altitudes 5-20 km.

Group II concludes autocorrelation functions of the meridional component of wind velocity with initial correlation levels higher than 20 km for the warm half-year in the middle and high latitudes.

Group III includes all autocorrelation functions of the meridional and zonal component of wind velocity in the cold half-year in the high latitudes, as well as the meridional component of wind velocity in the cold half-year in the middle latitudes with initial correlation levels higher than 20 km.

Group IV consists of autocorrelation functions of the zonal component of the wind velocity for the warm half-year in the middle and high latitudes, as well as for the cold half-year in the high latitudes above 20 km. /103

It is advantageous to approximate the functions in Group I with expression (4.1), and those in groups II, III, and IV -- with expression (4.2). Table 4.3 presents the coefficients in the exponents for these groups of functions obtained by the method of least squares.

TABLE 4.3. COEFFICIENTS OF EXPONENTS IN THE AUTOCORRELATION FUNCTIONS OF WIND VELOCITY COMPONENTS ($\Omega_0 = 0.08$ km^{-1})

Group of functions	I	II	III	IV
k, km^{-1}	0.100	0.190	0.055	0.0275

The precision of the analytic determination of these functions can be judged from the values of the root mean square errors of the approximation. They are given in Table 4.4 for the four groups of autocorrelation functions of wind velocity components.

The root mean square errors given in Table 4.4 have approximately the same value as the confidence intervals of the autocorrelations of wind velocity components.

In several cases, not only is the analytic form of the correlation functions of the physical parameters of the atmosphere of interest, but also the covariance functions of these parameters. The direct approximation in most cases is extremely difficult, since the correlation functions of the physical parameters of the

TABLE 4.4. GROUP MEAN SQUARE ERRORS OF THE APPROXIMATION OF AUTOCORRELATION FUNCTIONS OF WIND VELOCITY COMPONENTS

Group of functions	ΔH km						
	5	10	15	20	25	30	35
I	0.09	0.10	0.08	0.05	0.06	0.07	0.05
II	0.06	0.04	0.03	0.02	0.06	0.08	0.09
III	0.05	0.08	0.09	0.08	0.07	0.07	0.06
IV	0.08	0.09	0.09	0.11	0.09	0.09	0.08

atmosphere (excluding air temperature) very usually by 3-5 orders of increase in altitude intervals. However, these covariance functions could be described with analytic expressions if we are able to find these expressions for the root mean square deviations of the corresponding atmospheric parameters. The root mean square deviations of air temperature and wind velocity components change /104 very complexly with altitude, which hampers their approximation, though in principle it is possible, for example, by using the exponential orthonormalized functions examined below. The analytic expression of the root mean square deviation can be readily derived for air density. Actually, let us consider the function

$$l_\rho (H) = \frac{\sigma_\rho (H)}{\sigma_{\rho_0}}, \tag{4.3}$$

in which σ_0 is the root mean square deviation of air density at the group level, and σ_ρ (H) is its value at the altitude H if we consider the values of the function l_ρ(H) for these groups of latitudes and half-years, it turns out that its variation with altitude is described approximately by the exponential (Fig. 4.1)

$$l_\rho (H) = e^{-0.15H}. \tag{4.4}$$

Therefore,

$$\sigma_\rho (H) = \sigma_{\rho_0} e^{-0.15H}. \tag{4.5}$$

When determining this expression it was assumed that $\sigma_{\rho_0} = 50$ g/m^3, which corresponds to the mean of the root mean square deviation of air density at ground level.

Eq. (4.5) smoothes over several details of the altitude distribution of the root mean square deviations of air density in different latitudes and half-years, but closely describes the general trend of their variation with altitude. /105

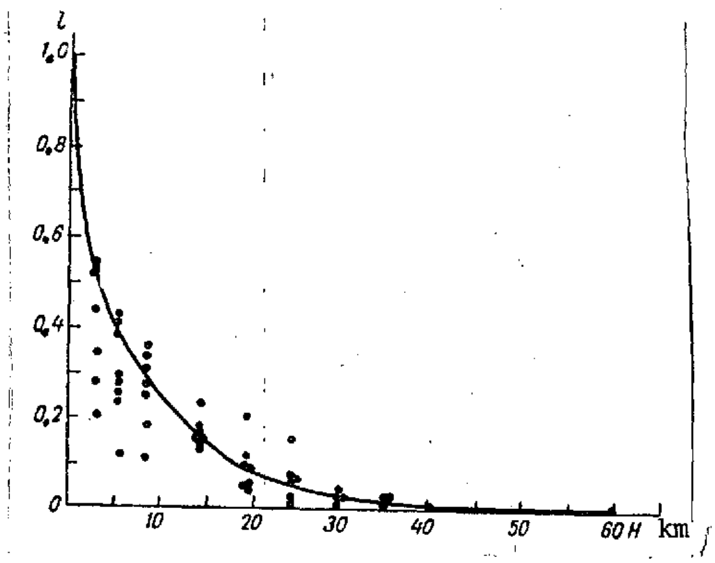


Fig. 4.1. Approximation of root mean square deviations of air density

Let us dwell on one feature of the fields of physical parameters of the atmosphere that follows from the properties of autocorrelation functions of air temperature and density and the wind velocity components. Obviously, these atmospheric parameters are nonstationary random functions of altitude. Let us consider the normalized function

$$\varphi(H) = \frac{\overset{\circ}{\xi}(H)}{\sigma_\xi(H)}, \qquad (4.6)$$

in which $\overset{\circ}{\xi}(H)$ are centered functions denoting either temperature or density of air, or else wind velocity component, and $\sigma_\xi(H)$ is the root mean square deviation.

Let us find the mathematical expectation, the covariance function, and the dispersion of this random function. It is obvious that

$$M[\varphi(H)] = 0, \quad R_\varphi(\Delta H) = r_\xi(\Delta H), \quad D_\varphi(H) = 1. \qquad (4.7)$$

Above it was shown that in each half-year in the middle and high latitudes this atmospheric layer can be divided into two layers, the first of which is the troposphere (the lower 20-km atmospheric layer is the first layer for the wind velocity components) in which the correlation functions of air temperature and density and wind velocity components, being covariance functions of the random functions $\phi(H)$ are identical in form, that is, their values are determined only by the altitude interval ΔH. Referring to Eqs.

(4.7), we can conclude that within these atmospheric layers the random functions $\phi(H)$ can be considered as stationary, while the obvious equality

$$\xi(H) = m_\xi(H) + \sigma_\xi(H)\varphi(H) \tag{4.8}$$

means that these nonstationary random functions (air temperature and density and wind velocity components) can be expressed in terms of nonrandom functions $m_\xi(H)$ and $\sigma_\xi(H)$ and by the stationary random function $\phi(H)$. The meridional and zonal wind velocity components in the cold half-year in the middle and high latitudes have the characteristic indicated above throughout the atmospheric layer extending from ground level to the altitude 70 km.

It must be noted that the approximation of the correlation function of the stationary random function by expression (4.2) involves one unpleasant feature, which is that the function (4.2) does not have a derivative at the point $\Delta H = 0$. However, this limitation is not essential for our purposes, since the small scales /106
of the fluctuations in the physical parameters of the atmosphere are not considered in the applications of this monograph.

In several practical applications it proves to be convenient to represent the autocorrelation functions of physical parameters of the atmosphere especially if they differ from the exponential, with a Fourier series in which the exponential orthonormalized functions are the basis functions.

The system of functions $y_1(x);\ y_2(x);\ \ldots;\ y_n(x),$

integrable on [a, b] is orthogonal if the scalar product of these functions satisfies the condition

$$(y_i, y_k) = \int_a^b y_i(x)\, y_k(x)\, dx = \begin{cases} 0 & \text{when } i \neq k \\ >0 & \text{when } i = k \end{cases} \ . \tag{4.9}$$

The system of orthonormalized functions

$$\psi_1(x);\ \psi_2(x);\ \ldots;\ \psi_n(x),$$

where

$$\psi_i(x) = \frac{y_i(x)}{\|y_i\|}, \tag{4.10}$$

and

$$\|y_i\| = (y_i\, y_i)^{1/2} = \left(\int_a^b y_i^2(x)\, dx\right)^{1/2} \tag{4.11}$$

is the norm of the function $y_i(x)$ can be brought into the correspondents with the system of functions.

The system of functions $\psi_i(x)$ has the following property:

$$(\psi_i, \psi_k) = \int_a^b \psi_i(x)\psi_k(x)\,dx = \begin{cases} 0 \text{ when } i \neq k \\ 1 \text{ when } i = k \end{cases}. \qquad (4.12)$$

The function f(x) integrable on [a, b] can be expanded in the Fourier series

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x), \qquad (4.13)$$

in which a_k are constants, and $\psi_k(x)$ are basis functions of the orthonormalized system.

The coefficients a_k are usually expressed in terms of f(x). Actually, let us multiply both sides of equality (4.13) by $\psi_n(x)$ and let us integrate the resulting product in the limits from a to b

$$\int_a^b f(x)\psi_n(x)\,dx = \sum_{k=1}^{\infty} a_k \int_a^b \psi_k(x)\psi_n(x)\,dx. \qquad (4.14)$$

By virtue of equality (4.12), all the integrals of the right side of Eq. (4.14) when $k \neq n$ tend to zero. Therefore, /107

$$\int_a^b f(x)\psi_n(x)\,dx = a_n \int_a^b \psi_n^2(x)\,dx = a_n. \qquad (4.15)$$

Equality (4.15) allows us to compute the coefficient of the Fourier series if we know the system of orthonormalized functions.

Suppose we have two vectors:

$$X:\ x_1;\ x_2;\ \ldots;\ x_p,$$
$$Y:\ y_1;\ y_2;\ \ldots;\ y_p.$$

Let us denote the smallest subspace X by S_X. The vectors X and Y will be equivalent if for all p

$$S_{x_p} \equiv S_{y_p}.$$

Let us orthogonalize these vectors, that is, let us replace the vector Y with some equivalent orthogonal vector. The orthogonization process amounts to the following.

If we project orthogonally x_p onto subspace S_{p-1}, we get

$$x_p = x_{pS_{p-1}} + x_{pN},$$

where

$$x_{pS_{p-1}} \in S_{p-1}; \quad ax_{pN} \perp S_{p-1} \quad (p = 1, 2, \ldots).$$

Let us set

$$y_p = \lambda_p x_{pN} \quad (p = 1, 2, \ldots, x_{1N} = x_1), \tag{4.16}$$

where λ_p are arbitrary nonzero quantities.

Then vector Y will be equivalent and orthogonal to vector X. However, we know /12/ that

$$x_{pN} = \frac{\begin{vmatrix} & & x_1 \\ & G & \vdots \\ & & x_{p-1} \\ (x_p, x_1) \ldots & (x_p, x_{p-1}) & x_p \end{vmatrix}}{G_{p-1}} \quad (p = 1, 2, \ldots, G_0 = 1), \tag{4.17}$$

where G = the Gram determinant.

Setting $\lambda = G_{p-1}$ in equality (4.16) and referring to equality (4.17), for the elements of the orthogonal vector we will have the following formulas: /108

$$y_1 = x_1; \quad y_2 = \begin{vmatrix} (x_1, x_1)\, x_1 \\ (x_2, x_1)\, x_2 \end{vmatrix}; \ \ldots;$$

$$y_p = \begin{vmatrix} (x_1, x_1) & \ldots & (x_1, x_{p-1})\, x_1 \\ \ldots & \ldots & \ldots \\ (x_p, x_1) & \ldots & (x_p, x_{p-1})\, x_p \end{vmatrix}, \tag{4.18}$$

in which as before the parentheses denote the scalar products of the functions appearing in them.

Now let us find the elements of the orthonormalized detector $\psi_p(x)$ corresponding to vector Y. To do this, let us determine the scalar product (y_p, y_p). If we use Eq. (4.16), we get

$$(y_p, y_p) = G^2_{p-1}(x_{pN}, x_{pN}).$$

It is known /12/ that

$$(x_{pN}, x_{pN}) = \frac{G_p}{G_{p-1}}. \tag{4.19}$$

Therefore,

$$(y_p, y_p) = G_p G_{p-1}.$$

Thus, the desired elements of the orthonormalized vector are determined by the equality

$$\psi_p = \frac{y_p}{(G_p G_{p-1})^{1/2}}. \tag{4.20}$$

Eq. (4.20) uniquely determines the orthonormalized system of vectors specified by Eq. (4.10).

To construct the orthonormalized system of exponential functions, let us assume that the vector X is given by the system

$$X(h);\quad e^{-ch};\quad e^{-2ch};\quad \ldots;\quad e^{-p\,ch}\quad (c>0) \tag{4.21}$$

and let us find the system of functions $\psi(h)$ corresponding to it. Suppose the scalar product of functions x_p with weight $g(x) = 1$ is determined at $[0, \infty]$. Then

$$y_p = \begin{vmatrix} \int\limits_0^\infty e^{-2ch}dh & \int\limits_0^\infty e^{-3ch}dh & \ldots & \int\limits_0^\infty e^{-p\,ch}\,dh & e^{-ch} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \int\limits_0^\infty e^{-(p+1)ch}dh & \int\limits_0^\infty e^{-(p+2)ch}dh & \ldots & \int\limits_0^\infty e^{-(2p-1)ch}dh & e^{-p\,ch} \end{vmatrix}$$

or, after computing the integrals in the determining determinant, /109

$$y_p = \begin{vmatrix} \frac{1}{2c} & \frac{1}{3c} & \cdots & \frac{1}{pc}e^{-ch} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{pc} & \frac{1}{(p+1)c} & \cdots & \frac{1}{(2p-2)c}e^{-(p-1)ch} \\ \frac{1}{(p+1)c} & \frac{1}{(p+2)c} & \cdots & \frac{1}{(2p-1)c}e^{-pch} \end{vmatrix}. \tag{4.22}$$

In this case, G_{p-1} is the algebraic complement of the element standing at the intersection of the (p+1)th row and column. Obviously,

$$G_{p-1} = \begin{vmatrix} \frac{1}{2c} & \frac{1}{3c} & \cdots & \frac{1}{pc} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{pc} & \frac{1}{(p+1)c} & \cdots & \frac{1}{2(p-1)c} \end{vmatrix} \tag{4.23}$$

and

$$G_p = \begin{vmatrix} \frac{1}{2c} & \frac{1}{3c} & \cdots & \frac{1}{pc} \\ \frac{1}{(p+1)c} & \frac{1}{(p+2)c} & \cdots & \frac{1}{(2p-1)c} \end{vmatrix}. \tag{4.24}$$

Computing the determinants (4.22) - (4.24), using Eq. (4.20) we can obtain a system of the specified number p of orthonormalized functions which consist of specific combinations of exponential functions:

$$\psi_k(h)=\sum_{\nu=1}^{k} B_{k\nu} e^{-\nu ch}. \tag{4.25}$$

Actually, when p = 1, Eqs. (4.22) - (4.24) yield

$$y_1(h)=e^{-ch};\quad G_0=1;\quad G_1=\int_0^\infty e^{-2ch}dh=\frac{1}{2c}.$$

Therefore,

$$\psi_1(h)=\sqrt{2c}\,e^{-ch}. \tag{4.26}$$

We similarly obtain, when p = 2,

$$y_2(h)=\begin{vmatrix} \frac{1}{2c} & e^{-ch} \\ \frac{1}{3c} & e^{-2ch} \end{vmatrix}=\frac{3e^{-2ch}-2e^{-ch}}{6c},\quad G_2=\begin{vmatrix} \frac{1}{2c} & \frac{1}{3c} \\ \frac{1}{3c} & \frac{1}{4c} \end{vmatrix}=\frac{1}{72c^2}$$

and /110

$$\psi_2(h)=\sqrt{c}\left(6e^{-2ch}-4e^{-ch}\right). \tag{4.27}$$

Similar computations yield

$$\psi_3(h)=\sqrt{6c}\left(10e^{-3ch}-12e^{-2ch}+3e^{-ch}\right); \tag{4.28}$$

$$\psi_4(h)=\sqrt{2c}\left(70e^{-4ch}-120e^{-3ch}+60e^{-2ch}-8e^{-ch}\right); \tag{4.29}$$

$$\psi_5(h)=\sqrt{10c}\left(126e^{-5ch}-280e^{-4ch}+210e^{-3ch}-60e^{-2ch}+5e^{-ch}\right). \tag{4.30}$$

These formulas coincide with the expressions obtained by another approach by D. Kh. Lening and R. G. Batten /32/.

Computing determinants of higher orders involves certain difficulties. Therefore let us examine another method of obtaining the orthonormalized system of exponential functions.

Let us set up the linear combination

$$A_1\psi_1+A_2\psi_2+\ldots+A_{n-1}\psi_{n-1}+A_n\psi_n,$$

in which coefficients A_n are given by the formula:

$$A_n=\int_0^\infty \psi_n e^{-nch}\,dh; \tag{4.31}$$

and let us determine the difference

$$K_n = A_n\psi_n - \sum_{l=1}^{n-1} A_l\psi_l. \tag{4.32}$$

Obviously, K_n is a polynomial of nth degree with respect to e^{-ch}.

In turn, ψ_n can be represented as

$$\psi_n = Q_n e^{-nch} + N_{n-1}, \tag{4.33}$$

where N_{n-1} is a polynomial of the (n-1)-th degree in e^{-ch}. Therefore we have

$$K_n = A_n Q_n e^{-nch} + A_n N_{n-1} - \sum_{l=1}^{n-1} A_l\psi_l. \tag{4.34}$$

Since $A_n Q_n = 1$, Eq. (4.34) becomes

$$L_n = e^{-nch} - \sum_{l=1}^{n} A_l\psi_l,$$

where

$$L_n = K_n - A_n N_{n-1}$$

for, if we consider equality (4.31),

$$L_n = e^{-nch} - \sum_{i=1}^{n-1} \psi_i(h) \int_0^\infty \psi_i(h)\, e^{-nch}\, dh \quad (n = 1, 2, \ldots). \tag{4.35}$$

Comparing polynomial $L_n(h)$ and $y_n(h)$ shows that they differ from each other by a cofactor that depends on n. From this it follows that /111

$$\psi_n(h) = \frac{L_n(h)}{\left[\int_0^\infty L_n^2(h)\, dh\right]^{1/2}}. \tag{4.36}$$

Eqs. (4.35) and (4.36) are a set of recursion formulas enabling us to easily calculate the sequence of orthonormalized exponential functions of any degree.

The sequence of coefficients $B_{\nu k}$ can also be found by another method proposed by A. S. Galkin and L. A. Mayboroda. It consists of the following. If we use Eqs. (4.12) and (4.25), we get the relation

$$\int_0^\infty \left(\sum_{\nu=1}^{k} B_{\nu k} e^{-\nu ch}\right)\left(\sum_{i=1}^{m} B_{im} e^{-ich}\right) dh = \sum_{\nu=1}^{k}\sum_{i=1}^{m} \frac{B_{k\nu}B_{mi}}{(k+i)c} = \begin{cases} 1 \text{ when } k = m, \\ 0 \text{ when } k \neq m. \end{cases} \tag{4.37}$$

Eq. (4.37) enables us to write out the system of algebraic equations to determine the coefficients of Eq. (4.25). In general, when there is a large number of terms in the series (4.25) the system of equations is complex. However, if we analyze the sequence of equation (4.34) we can note that the coefficient $B_{k\nu}$ are defined by the expression

$$B_{k\nu} = (-1)^{k+\nu}\sqrt{2ck}\,\frac{(k+\nu-1)!}{\nu!\,(\nu-1)!\,(k-\nu)!}. \tag{4.38}$$

Here the norm of function (4.35) is

$$\|L_k\| = \frac{k!\,(k-1)!}{\sqrt{2kc}\,(2k-1)!}.$$

When computing the coefficients $B_{k\nu}$ by this method, we can use the following recursion relations:

$$\left.\begin{aligned} B_{k+1,\nu} &= -\frac{\sqrt{2k+2}\,(k+\nu)}{(k-\nu+1)\sqrt{2k}}\,B_{k\nu}, \\ B_{k,\nu+1} &= -\frac{(k+\nu)(k-\nu)}{\nu(\nu+1)}\,B_{k\nu}. \end{aligned}\right\} \tag{4.39}$$

Obtaining the system of orthonormalized function (4.25) we can approximate the function f(h) with the series

$$f(h) = \sum_{n=1}^{N} S_n \psi_n(h), \tag{4.40}$$

in which /112

$$S_n = \int_0^\infty f(h)\,\psi_n(h)\,dh,$$

and the integral quadratic error of the approximation is

$$\Delta = \int_0^\infty \left[f(h) - \sum_{n=1}^{N} S_n \psi_n(h)\right]^2 dh \tag{4.41}$$

or after several transformations

$$\Delta = \int_0^\infty f^2(h)\,dh - \sum_{n=1}^{N} S_n^2.$$

When approximating the functions f(h) with orthogonal exponential polynomials, owing to the finite number of the polynomials the problem of selecting the value of parameter c crops up. One of the approaches to the solution of c has been proposed in [32]; accordingly, the following sequence in selecting the coefficients c is recommended:

subtract, if necessary, from function f(h) a constant so that the resulting function $\Delta f(h) \to 0$ as $h \to \infty$; and

select the appropriate value of c so that the function e^{-ch} tends to zero at approximately the same rate as the function $\Delta f(h)$.

However, this selection of the value of the coefficient c cannot be regarded as rigorous since the number of terms in the approximating series depend not only on the specified precision of the approximation, but also on the value of $\Delta(c)$.

To obtain the value of c corresponding to $\Delta(c) = \min$, it is convenient to use numerical methods. In particular, we can use methods whose description is given in [71]. It is also to use the following iterative approach. Let us expand the function $\Delta(c)$ in a Taylor series in the parameter c in the neighborhood of some selected initial value of c_0 bounded by the quadratic approximation

$$\Delta(c) = \Delta(c_0) + \frac{\partial \Delta}{\partial c}(c_0)(c - c_0) + \frac{1}{2}\frac{\partial^2 \Delta}{\partial c^2}(c_0)(c - c_0)^2. \tag{4.42}$$

The minimum value of Eq. (4.42) is obtained when

$$\frac{\partial \Delta}{\partial c}(c_0) + \frac{\partial^2 \Delta}{\partial c^2}(c_0)(c - c_0) = 0.$$

Hence the revised value of $c = c_1$ is

$$c_1 = c_0 - \frac{\frac{\partial \Delta}{\partial c}(c_0)}{\frac{\partial^2 \Delta}{\partial c^2}(c_0)}. \tag{4.43}$$

Further, in Eqs. (4.42) and (4.43), instead of c_0 we substitute [113] c_1 and we obtain c_2 even closer to the optimal value. The sequence $c_0, c_1, c_2, \ldots$ tends to c, where $\Delta c = \min$.

Extension (4.40), after the reduction of similar terms, is transformed to become

$$f(h) = \sum_{\mu=1}^{N} w_\mu e^{-\mu ch}. \tag{4.44}$$

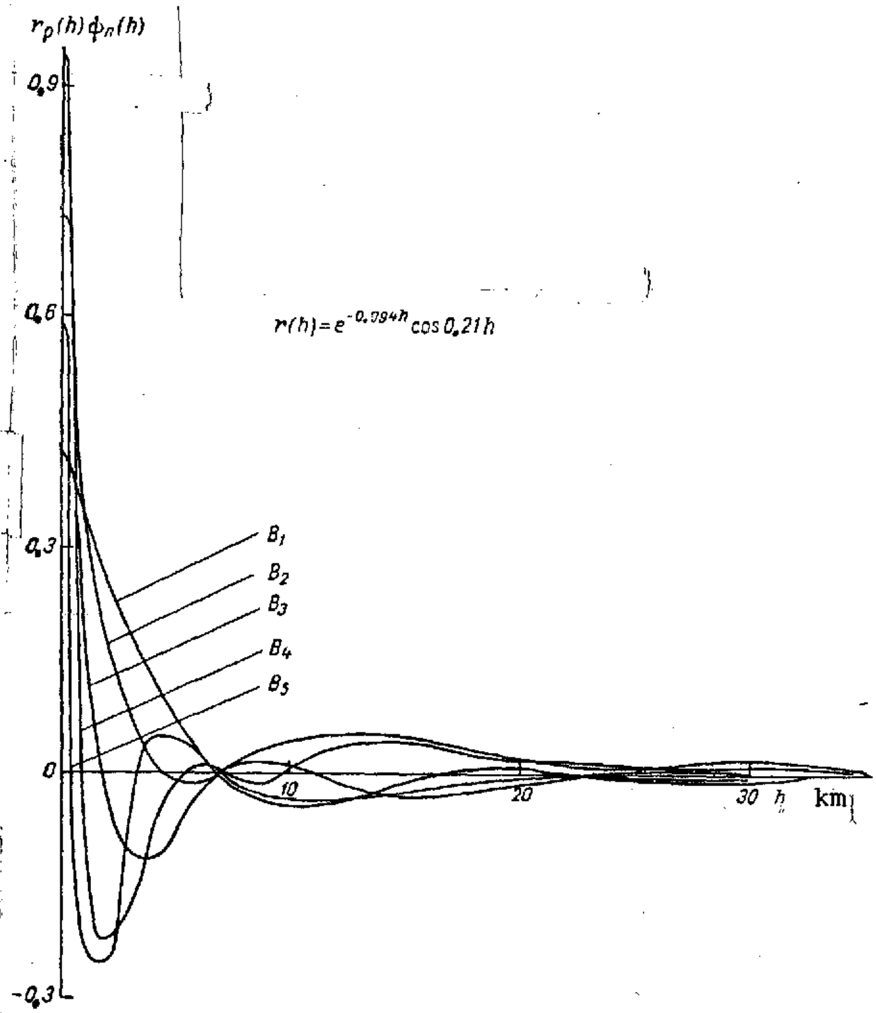


Fig. 4.2. Values of the integrand in Eq. (4.15)

Now let us assume that $f(h) = r_\xi(\Delta H)$ and let us find the expansions of the autocorrelation functions of air temperature and density that refer to the first two groups of functions, that is, to groups differing from the exponential. Calculations show that a fairly close approximation is attained for these functions if we use only the five terms of series (4.44). To do this, as indicated above, it is sufficient to determine five values of the coefficients by Eq. (4.15). As applied to group II of the normalized autocorrelation functions of air temperature and density, the values of the integrands of Eq. (4.15) are given in Fig. 4.2.

After computing the coefficients of the Fourier series, S_n, and reducing related terms, we get autocorrelation functions of air temperature and density in the form /114

$$r_{\xi}(\Delta H) = \sum_{\mu=1}^{5} w_{\mu} e^{-\mu c h}. \qquad (4.45)$$

The values of coefficients w_{μ} and c are given in Table 4.5.

TABLE 4.5. VALUES OF THE COEFFICIENTS w_{μ} AND c OF THE EXPANSION OF THE AUTOCORRELATION FUNCTIONS OF TEMPERATURE AND DENSITY IN A FOURIER SERIES

Group of functions	c	w_{μ} for				
		$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
I	0,040	−1,820	30,160	−121,101	168,060	−74,280
II	0,094	1,691	−22,146	62,092	−60,358	19,740

As follows from equality (4.45), when $\Delta H = 0$, we have

$$r_{\xi}(\Delta H = 0) = \sum_{\mu=1}^{5} w_{\mu} = 1.$$

Summing up the coefficients in Table 4.5 gives the values of $r_{\xi}(\Delta H = 0)$ very close to unity.

4.2. Noncanonical Expansions of Temperature and Air Density and Wind Velocity Components

When solving several problems of analyzing and synthesizing automatic control systems, a noncanonical expansion of some stationary random function $\xi(h)$ refers to an expansion of the form /79/

$$\xi(h) = m_{\xi}(h) + \gamma\cos\Omega^* h + \beta\sin\Omega^* h, \qquad (4.46)$$

in which γ and β are independent, normally distributed random variables, and Ω^* is a random frequency that has some distribution with probability density $p(\Omega)$. Random variables γ and β have the following property:

$$\left.\begin{array}{l} M[\gamma] = M[\beta] = 0; \quad M[\gamma\beta] = 0; \\ M[\gamma^2] = M[\beta^2] = D_{\gamma} = D_{\beta} = D. \end{array}\right\} \qquad (4.47)$$

Normalized centered functions $\phi_\xi(h)$, where $\xi(h)$ denotes air temperature and density and wind velocity components, as shown in Section 4.1, can be approximately assumed to be stationary random functions (we can judge the precision of this approximation from the ratio between the root mean square error of the approximation of the covariance functions of each of the groups of these functions by an analytic expression and by the confidence intervals of the covariance functions). Accordingly, let us examine noncanonical expansions of air temperature and density and wind velocity components /115

$$\varphi_t(h)=\frac{t(h)-m_t(h)}{\sigma_t(h)}=\gamma_t\cos\Omega_t^*h+\beta_t\sin\Omega_t^*h, \tag{4.48}$$

$$\varphi_\rho(h)=\frac{\rho(h)-m_\rho(h)}{\sigma_\rho(h)}=\gamma_\rho\cos\Omega_\rho^*h+\beta_\rho\sin\Omega_\rho^*h, \tag{4.49}$$

$$\varphi_u(h)=\frac{u(h)-m_u(h)}{\sigma_u(h)}=\gamma_u\cos\Omega_u^*h+\beta_u\sin\Omega_u^*h, \tag{4.50}$$

$$\varphi_v(h)=\frac{v(h)-m_v(h)}{\sigma_v(h)}=\gamma_v\cos\Omega_v^*h+\beta_v\sin\Omega_v^*h. \tag{4.51}$$

As indicated above, random variables γ and β are distributed according to the normal law, that is,

$$p(\gamma)=\frac{1}{\sqrt{2\pi D}}e^{-\frac{\gamma^2}{2D}}, \tag{4.52}$$

$$p(\beta)=\frac{1}{\sqrt{2\pi D}}e^{-\frac{\beta^2}{2D}}. \tag{4.53}$$

Now let us find the distribution of random variables Ω^*. To do this, let us use the equality

$$R_\varphi(\Delta h)=M[\varphi(h)\varphi(h+\Delta h)]=r_\xi(\Delta h), \tag{4.54}$$

which is one of the sets of equalities (4.7). Let us substitute into Eq. (4.54) instead of $\phi(h)$, their values from Eqs. (4.48) - (4.51). Then referring to Eqs. (4.47), we get

$$r_\xi(\Delta h)=DM(\cos\Omega_\xi^*\Delta h).$$

Since D = 1 (this is evident if in the right-hand expression we set $\Delta h = 0$), then

$$r_\xi(\Delta h)=M(\cos\Omega_\xi^*\Delta h)$$

or

$$r_\xi(\Delta h)=\int_{-\infty}^{\infty}p(\Omega)\cos\Omega\Delta h\,d\Omega. \tag{4.55}$$

The inverse transformation for Eq. (4.55) is of the form

$$p(\Omega)=\frac{1}{\pi}\int_0^{\infty}r_\xi(\Delta h)\cos\Omega\Delta h\,d(\Delta h). \tag{4.56}$$

In Section 4.1 it was shown that autocorrelation functions of air temperature and density have the form of the exponential (4.2) for stratospheric functions, and the form of a damping cosine (4.1) for tropospheric functions, and that the autocorrelation functions of wind velocity components are determined by Eq. (4.2). Substituting Eqs. (4.1) and (4.2) into Eq. (4.56) and integrating it, we get a distribution of the form /116

$$p(\Omega)=\frac{k}{\pi(k^2+\Omega^2)} \tag{4.57}$$

For random variables Ω^* of noncanonical expansions of air temperature and density in the stratosphere and mesosphere, and of wind velocity components, as well as the distribution

$$p(\Omega)=\frac{k}{\pi}\left[\frac{1}{k^2+(\Omega-\Omega_0)^2}+\frac{1}{k^2+(\Omega+\Omega_0)^2}\right], \tag{4.58}$$

referring to air temperature and density in the troposphere. The classification of Eqs (4.57) and (4.58) in a particular physical parameter of the atmosphere, half-year, and latitude is determined by the values of the coefficients k and Ω.

Distributions (4.57) - (4.58) can be expressed in terms of a normal distribution. To do this, let us examine the two-parameter distribution

$$p(\eta)=\frac{1}{2\pi}\left[e^{-\frac{(\eta-\eta_0)^2}{2}}+e^{-\frac{(\eta+\eta_0)^2}{2}}\right], \tag{4.59}$$

in which we denote

$$\eta-\eta_0=x_1;\ \eta+\eta_0=x_2.$$

Then distribution (4.58) reduces to the distribution

$$p(\eta)=p(x_1,x_2)=\frac{1}{2\pi}\left(e^{-\frac{x_1^2}{2}}+e^{-\frac{x_2^2}{2}}\right)=p(\Omega) \tag{4.60}$$

given the condition that

$$x_1=\left(2\ln\frac{k^2+(\Omega-\Omega_0)^2}{k\sqrt{\frac{2}{\pi}}}\right)^{1/2} \tag{4.61}$$

and

$$x_2=\left(2\ln\frac{k^2+(\Omega+\Omega_0)^2}{k\sqrt{\frac{2}{\pi}}}\right)^{1/2}. \tag{4.62}$$

Actually, if we substitute Eqs. (4.61) and (4.62) into distribution (4.59), we will have /117

$$p(\eta) = \frac{1}{\sqrt{2\pi}}\left[e^{\ln\frac{k\sqrt{\frac{2}{\pi}}}{k^2+(\Omega-\Omega_0)^2}} + e^{\ln\frac{k\sqrt{\frac{2}{\pi}}}{k^2+(\Omega+\Omega_0)^2}}\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\left(\sqrt{\frac{2}{\pi}}\frac{k}{k^2+(\Omega-\Omega_0)^2} + \sqrt{\frac{2}{\pi}}\frac{k}{k^2+(\Omega+\Omega_0)^2}\right) =$$

$$= \frac{k}{\pi}\left[\frac{1}{k^2+(\Omega-\Omega_0)^2} + \frac{1}{k^2+(\Omega+\Omega_0)^2}\right] = p(\Omega).$$

In the particular case when in Eq.(4.59) $\eta_0 = 0$, we have the distribution

$$p(\eta) = \sqrt{\frac{2}{\pi}}e^{-\frac{\eta^2}{2}} = p(\Omega), \qquad (4.63)$$

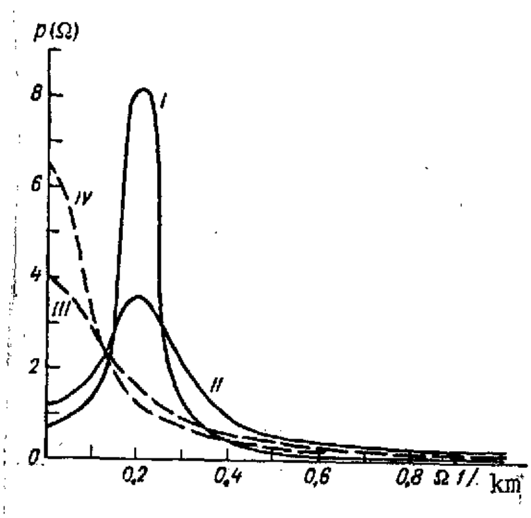
to which the distribution (4.57) reduces if

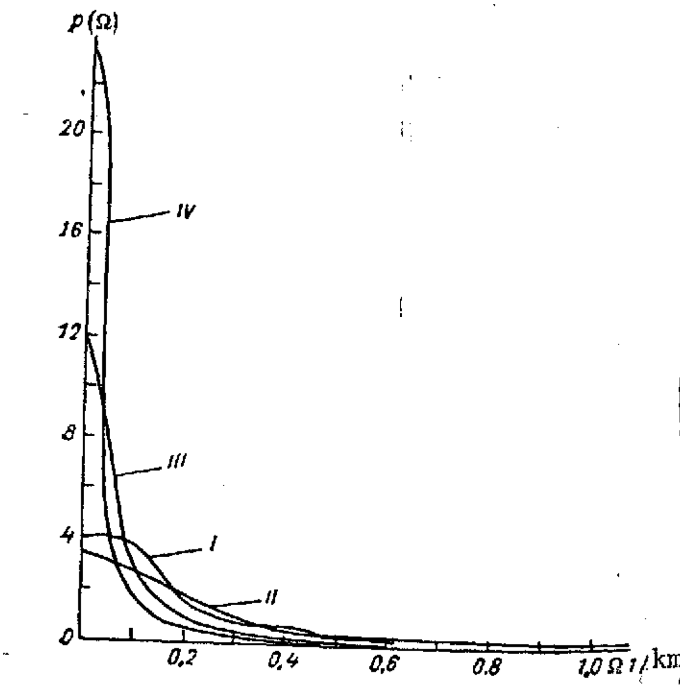
$$\eta = \left(2\ln\frac{\sqrt{2\pi}(k^2+\Omega^2)}{k}\right)^{1/2}. \qquad (4.64)$$

Distributions (4.60) and (4.63) are more convenient compared with distributions (4.57) - (4.58) when we use, for example, the interpolational method of analyzing automatic control systems, which will be examined below, since in this case the distributions of all random variables of the noncanonical expansion of atmospheric perturbations are of the same form.

The distributions of frequencies of fluctuations in air temperature and density and in wind velocity components, determined by Eqs. (4.57) and (4.58), are expressed in Figs. (4.3) and (4.4). The numbers alongside the curves correspond to specific groups of autocorrelational functions of the physical parameters of the atmosphere, that is, they indicate the particular layer of the atmosphere, each latitudinal group, and each half-year to which each function belongs (see Section 4.1). The laws of the distribution of frequency make it possible to determine which scales of fluctuations in air temperature and density, zonal and meridional wind velocity components prevail in specific atmospheric layers (we have in mind the fluctuations of the meso-scale). To do this, we must compute the probability P that frequencies of atmospheric perturbations Ω will fall in a specified interval of frequencies

$$p\left(\Omega_{i-1} \leqslant \Omega < \Omega_i\right) = \int_{\Omega_{i-1}}^{\Omega_i} P(\Omega)\, d\Omega. \qquad (4.65)$$





/118

Fig. 4.3. Distribution of frequencies of perturbations of air density

Fig. 4.4. Distribution of frequencies of perturbations of wind velocity components

Computing the probability P by means of Eq. (4.65) reduces to substituting in it Eqs. (4.57) and (4.58) and then integrating in the specified frequency interval. The integration gives, respectively, /119

$$P(\Omega_{i-1} \leqslant \Omega < \Omega_i) = \frac{2}{\pi} \operatorname{arctg} \frac{\Omega}{k} \Bigg|_{\Omega_{i-1}}^{\Omega_i}, \qquad (4.66)$$

$$P(\Omega_{i-1} \leqslant \Omega < \Omega_i) = \frac{2}{\pi} \left[\operatorname{arctg} \frac{\Omega - \Omega_0}{k} + \operatorname{arctg} \frac{\Omega + \Omega_0}{k}\right] \Bigg|_{\Omega_{i-1}}^{\Omega_i}. \qquad (4.67)$$

The results of computing the probabilities of the frequencies of fluctuations in air density are given in Table 4.6.

TABLE 4.6. PROBABILITIES OF FREQUENCIES OF FLUCTUATIONS IN AIR DENSITY

Layer, km	Latitude	Ω_{i-1} – Ω_i km^{-1}					
		0,1	0,1–0,2	0,2–0,3	0,3–0,4	0,4–0,5	0,5–0,6
10–60	Middle & high	0,57	0,13	0,09	0,05	0,03	0,03
0–60	Middle	0,07	0,32	0,44	0,08	0,03	0,01
0–60	High	0,13	0,26	0,29	0,12	0,06	0,03

From Table 4.6 it follows that in the stratosphere and mesosphere (10-60 km layer) in the middle and high latitudes in the warm and cold half-years, the highest probability is noted for the smallest (less than 0.1 km^{-1}) frequencies of fluctuations in air density. The probability of higher frequencies (smaller wavelength) drops off sharply. For example, the frequency intervals 0.5-0.6 km^{-1} (wavelength 12.5-10.5 km) has the probability 0.03. If we examine, in addition to the stratosphere and mesosphere, also the troposphere, the maximum probability shifts toward the side of higher frequencies. In the middle latitudes, for example, the highest probability falls in the 0.2-0.3 km^{-1} interval (wavelength 31-21 km). The same shift in probabilities along the frequency spectrum is observed also in the high latitudes, although it proves to be more elongated there.

Table 4.7 contains the probabilities of the frequencies of fluctuations in the zonal and meridional wind velocity components (see Section 4.1 for the symbols of the groups of functions).

As follows from Table 4.7, for the zonal wind velocity components in the warm half-year in the middle and high latitudes and in the warm half-year in the high latitudes (IV), frequencies less than 0.1 km^{-1} are prevalent, absolutely. The greatest probability, although still somewhat smaller in value, occurs at the same frequency also for the meridional and zonal wind velocity components in the cold half-year in the high latitudes (III). A more uniform shift of probabilities along the frequency spectrum occurs for the meridional component in the cold half-year in the middle latitudes (I) and in the warm half-year in the high latitudes (II).

TABLE 4.7. PROBABILITIES OF FREQUENCIES OF FLUCTUATIONS IN ZONAL AND MERIDIONAL WIND VELOCITY COMPONENTS /120

Group of functions	$\omega_{i-1} \div \omega_i$ km^{-1}					
	0,1	0,1–0,2	0,2–0,3	0,3–0,4	0,4–0,5	0,5–0,6
I	0,40	0,27	0,11	0,06	0,03	0,02
II	0,31	0,20	0,13	0,08	0,05	0,04
III	0,68	0,15	0,06	0,03	0,02	0,01
IV	0,83	0,09	0,03	0,02	0,01	0,01

4.3. Canonical Expansions of Air Temperature and Density and of Wind Velocity Components

In several cases it is convenient to represent the random function as a certain linear combination of uncorrelated random variables. This combination contains nonrandom functions and is of the following form:

$$\xi(t) = m_\xi(t) + \sum_{\nu=1}^{\infty} v_\nu x_\nu(t). \tag{4.68}$$

In Eq. (4.68) we have designated: $x_\nu(t)$ are nonrandom functions, called coordinate functions, and v_ν are uncorrelated random variables such that

$$M[v_\nu] = 0$$
$$M[v_\nu v_\mu] = \begin{cases} 0 & \text{for } \nu \neq \mu \\ D_\nu & \text{for } \nu = \mu. \end{cases} \tag{4.69}$$

The representation of a random function in the form (4.68) is customarily called the canonical expansion of the random function. In the general case, the canonical expansion of a random function is an infinite series. When practical use is made of canonical expansions, ordinarily a limited number of terms in the series (4.68) are employed.

If we denote a centered canonical function by $\overset{\circ}{\xi}(t)$, then obviously the covariance function of the random variable $\xi(t)$ is

$$R_\xi(t, t') = M\left[\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t')\right]. \tag{4.70}$$

Referring to Eq. (4.68), we can rewrite Eq. (4.70) in the form /121

$$R_\xi(t, t') = M\left[\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty} v_\nu v_\mu x_\nu(t) x_\mu(t')\right].$$

In the last equality we used the property of the interchangeability of the mathematical expectation, and also the property of random coefficients of the canonical expansion of random function (4.69). We will have

$$R_\xi(t, t') = \sum_\nu D_\nu x_\nu(t) x_\nu(t'). \tag{4.71}$$

Eq. (4.71) is the canonical expansion of the covariance functions of the random function $\xi(t)$. The equality

$$D_\xi(t) = R_\xi(t, t') = \sum_\nu D_\nu [x_\nu(t)]^2, \tag{4.72}$$

is its particular case when $t = t'$, where $D_\xi(t)$ is the dispersion of the random function $\xi(t)$.

To determine the coordinate functions of the canonical expansion of the random function, let us find the mathematical expectation of the product /49

$$M\left[\overset{\circ}{\xi}(t)v_\mu\right]=\sum_\nu M\left[v_\nu v_\mu\right]x_\nu(t). \tag{4.73}$$

Based on Eqs. (4.69), all the terms in the right-hand side of Eq. (4.73) are equal to zero, with the exception of the term for which $\nu=\mu$. Thus,

$$M\left[\overset{\circ}{\xi}(t)v_\mu\right]=D_\mu x_\mu(t),$$

which yields

$$x_\mu(t)=\frac{1}{D_\mu}M\left[\overset{\circ}{\xi}(t)v_\mu\right]. \tag{4.74}$$

In order to determine the random coefficients of the canonical expansions of the random function, let us represent them in the form of a linear combination of values of these centered random function $\overset{\circ}{\xi}(t)$

$$v_\nu=\sum_h a_{\nu h}\overset{\circ}{\xi}(t_h), \tag{4.75}$$

in which $a_{\nu h}$ are arbitrary coefficients.

The random coefficients v_ν can be found if we know from the values of $a_{\nu h}$. The coefficients $a_{\nu h}$ are not difficult to determine by using the following obvious equality:

$$M\left[v_\nu v_\mu\right]=\sum_{h,l}a_{\nu h}a_{\mu l}R_\xi(t_h,t_l). \tag{4.76}$$

When $\nu\neq\mu$, as follows from Eq. (4.69), the left-hand side of Eq. (4.76) is equal to zero, that is, /122

$$\sum_{h,l}a_{\nu h}a_{\mu l}R_\xi(t_h,t_l)=0. \tag{4.77}$$

The coefficients $a_{\nu h}$ that satisfies the conditions (4.77) can be selected, and by using an infinite set of methods, since the number of Eqs. (4.77) is always smaller than the number of coefficients. For example, first we can determine the random variable v_1 by arbitrarily specifying the coefficients a_{1h}, and then we can find the coefficients a_{2h} so that the random variable v_2 will not be correlated with v_1. Further, we can determine all the remaining coefficients a_{nh} such that the random value v_n is not correlated with the variables $v_1, v_2, \ldots, v_{n-1}$. Thus, all the coefficients a_{nh} can be specified arbitrarily, except for the coefficient $a_{(n-1)h}$. For example, we can specify

$$a_{\nu\nu}=1;\ a_{\nu h}=0 \text{ when } h\geqslant\nu. \tag{4.78}$$

Then the remaining coefficients $a_{\nu h}$ (for $h < \nu$) will be computed by means of Eqs. (4.77). Consequently, referring to Eqs. (4.78), Eq. (4.75) will become:

$$\left.\begin{aligned} v_1 &= \overset{\circ}{\xi}(t_1), \\ v_\nu &= \sum_{h=1}^{\nu-1} a_{\nu h}\overset{\circ}{\xi}(t_h) + \overset{\circ}{\xi}(t_\nu) \quad (\nu = 2, 3, \ldots). \end{aligned}\right\} \tag{4.79}$$

Setting $\nu = \mu$, in Eq. (4.76), we can determine the dispersion D_ν. It will be expressed by the equality

$$D_\nu = M[v_\nu^2] = \sum_{h,l} a_{\nu h} a_{\nu l} R_\xi(t_h, t_l). \tag{4.80}$$

If, moreover, in Eq. (4.74) we substitute the expression (4.75), we will have

$$x_\nu(t) = \frac{1}{D_\nu} \sum_h a_{\nu h} R_\xi(t_1, t_h). \tag{4.81}$$

Now, to determine all the elements of the canonical expansion of the random function, let us write out first Eqs. (4.79) and let us compute the values of the dispersion and of the coordinate function for $\nu = 1$. We get

$$v_1 = \overset{\circ}{\xi}(t_1); \; D_1 = R_\xi(t_1, t_1) \quad \text{and} \quad x_1(t) = \frac{1}{D_1} R_\xi(t_1, t_1), \tag{4.82}$$

and let us determine the variables v_ν from the recursion formulas /123 deriving from (4.79)

$$v_\nu = \sum_{h=1}^{\nu-1} c_{\nu h} v_h + \overset{\circ}{\xi}(t_1). \tag{4.83}$$

In order to find the coefficients $c_{\nu h}$, let us set $t = t_\nu$ in (4.68) and let us compare the resulting expression with Eq. (4.83). We will have

$$c_{\nu\mu} = -x_\mu(t_\nu) \quad \begin{pmatrix} \mu = 1, 2, \ldots, \nu - 1 \\ \nu = 2, 3, \ldots \end{pmatrix} \tag{4.84}$$

and, moreover,

$$x_\nu(t_\nu) = 1 \quad (\nu = 1, 2, \ldots), \tag{4.85}$$

and also

$$x_\mu(t_\nu) = 0 \text{ for } \mu > \nu. \tag{4.86}$$

Based on Eqs. (4.84), Eq. (4.83) can be written in the form

$$v_\nu = \overset{\circ}{\xi}(t_\nu) - \sum_{h=1}^{\nu-1} v_h x_h(t_h) \quad (\nu = 2, 3, \ldots). \tag{4.87}$$

From Eq. (4.87) there derives an equation for determining the dispersions

$$D_\nu = R_\xi(t_\nu, t_\nu) - \sum_{h=1}^{\nu-1} D_h [x_h(t_\nu)]^2 \quad (\nu = 2, 3, \ldots). \tag{4.88}$$

Now it remains to write out in its final form the equation for determining the coordinate functions. For this purpose, let us substitute Eq. (4.87) into Eq. (4.74). With reference to Eq. (4.69), we get

$$x_\nu(t) = \frac{1}{D_\nu}\left[R_\xi(t, t_\nu) - \sum_{h=1}^{\nu-1} D_h x_h(t) x_h(t_\nu)\right] (\nu = 2, 3, \ldots). \tag{4.89}$$

Eqs. (4.87) - (4.88), joined with the first equality from (4.82), constitute a system of formulas that enables us to determine successively all the elements of the noncanonical expansion of the random function $\xi(t)$.

If as before we consider the normalized centered functions of the physical parameters of the atmosphere

$$\varphi_\xi(H) = \frac{\xi(H) - m_\xi(H)}{\sigma_\xi(H)}, \tag{4.90}$$

the canonical expansion of the correlation functions of which are of the form

$$r_\xi(H, H') = \sum_\nu D_\nu^* x_\nu(H) x_\nu(H'), \tag{4.91}$$

then the formulas for computing the elements of the canonical expansion of these random functions will be of the following form: /124

$$\begin{gathered} w_1 = \overset{\circ}{\varphi}_\xi(H_1), \\ w_\nu = \overset{\circ}{\varphi}_\xi(H_\nu) - \sum_{i=1}^{\nu-1} w_i x_i(H_i) \quad (\nu = 2, 3, \ldots); \\ D_\nu^* = 1 - \sum_{i=1}^{\nu-1} D_i^* [x_i(H_\nu)]^2; \\ x_\nu(H) = \frac{1}{D_\nu^*}\left[r_\xi(H, H_\nu) - \sum_{i=1}^{\nu-1} D_i^* x_i(H) x_i(H_\nu)\right]. \end{gathered} \tag{4.92}$$

TABLE 4.8. ELEMENTS OF CANONICAL EXPANSION OF AIR TEMPERATURE, COLD HALF-YEAR, MIDDLE LATITUDES

i	$\mathring{\varphi}_{\rho}(H)$														$w_{\rho i}$	$D^{*}_{\rho i}$
	0,20	0,20	0,15	0,20	0,10	0,25	0,50	0,80	0,20	0,35	0,40	0,30	0,20	0,12		
	H km															
	3	6	9	15	20	25	30	35	40	45	50	55	60	65		
1	1	0,3530	–0,0670	–0,7350	–0,3310	0,0190	0,0730	0,1530	–0,0810	–0,0310	–0,0810	0,0010	0,0310	0,0200	0,2000	1,0000
2	0	1	0,0167	–0,2428	–0,1452	–0,0671	0,0779	–0,0251	0,0372	–0,0572	–0,1421	–0,0175	–0,0011	0,0148	0,1294	0,8754
3	0	0	1	0,2655	0,0914	0,0455	–0,0083	–0,0084	–0,0361	–0,0143	–0,0265	0,0506	0,0122	0,0112	0,1612	0,9953
4	0	0	0	1	0,5583	0,7032	0,6432	0,5602	–0,0831	–0,0952	0,0836	0,3895	0.6489	0,3399	0.3356	0,3380
5	0	0	0	0	1	0,3494	0,1720	0,1826	0,0935	–0,0201	–0,1401	–0,2637	0,0705	–0,0220	–0,0171	0,7583
6	0	0	0	0	0	1	0,1486	–0,0035	0,0413	0,0107	–0,0468	–0,1363	–0,3226	–0,1839	0,0175	0,7339
7	0	0	0	0	0	0	1	0,3888	0,2645	0,1581	0,0562	–0,0800	–0.3315	–0,1645	0,2611	0,8108
8	0	0	0	0	0	0	0	1	0,6612	0,4118	0,2947	0,1351	0,0359	–0.1697	0,4877	0,7221
9	0	0	0	0	0	0	0	0	1	0,4813	0,4226	0,3681	0,3451	0,2211	–0,1454	0,6083
10	0	0	0	0	0	0	0	0	0	1	0,5197	0,3975	0,2740	0,1980	0,2252	0,7088
11	0	0	0	0	0	0	0	0	0	0	1	0,5068	0,3447	0,2528	0,1952	0,5908
12	0	0	0	0	0	0	0	0	0	0	0	1	0,4174	0,3050	–0,0188	0,5150
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0,9116	0,0267	0,4008
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0,0764	0,4160

TABLE 4.9. ELEMENTS OF CANONICAL EXPANSION OF AIR TEMPERATURE. COLD HALF-YEAR, MIDDLE LATITUDES

t	$\overset{\circ}{\varphi}_t(H)$														w_{tt}	D^*_{tt}
	1,8	1,6	1,2	1,1	0,8	0,8	0,6	0,6	0,4	0,6	0,6	1,1	1,3	1,2		
	H km															
	3	6	9	15	20	25	30	35	40	45	50	55	60	65		
1	1	0,6850	0,6410	−0,4390	0,2950	0,1400	0,0480	0,0870	−0,0400	−0,0680	−0,027	−0,0100	0,0300	0,0500	1,800	1,000
2	0	1	−0,6875	−0,5921	−0,1961	0,0737	0,1151	−0,0482	−0,0445	−0,1534	−0,0292	−0,0248	0,0178	0,0108	0,3670	0,5308
3	0	0	1	−0,1411	0,0160	0,0266	0,0526	−0,0669	−0,0918	−0,1565	−0,1274	−0,0292	0,0425	0,0167	0,2985	0,8400
4	0	0	0	1	0,5042	−0,0483	0,1480	0,1292	0,2358	0,1743	0,1751	0,0899	0,0723	0,0617	2,1496	0,6045
5	0	0	0	0	1	0,3300	0,2125	0,1392	0,1564	0,0272	0,0477	0,0316	0,0021	−0,0171	−0,7476	0,7387
6	0	0	0	0	0	1	0,6626	0,5419	0,3566	0,2620	0,1948	0,0871	0,0421	0,0051	0,8635	0,8951
7	0	0	0	0	0	0	1	0,5068	0,1959	0,1894	0,1193	0,0768	0,0747	0,0404	−0,2758	0,5030
8	0	0	0	0	0	0	0	1	0,4567	0,4683	0,3467	0,1865	0,0177	−0,0157	−0,0208	0,5710
9	0	0	0	0	0	0	0	0	1	0,2057	0,2252	0,1694	0,0909	0,0950	−0,1186	0,6864
10	0	0	0	0	0	0	0	0	0	1	0,5086	0,4141	0,3603	0,1917	0,3313	,7096
11	0	0	0	0	0	0	0	0	0	0	1	0,3815	0,2319	0,2581	0,0801	0,6221
12	0	0	0	0	0	0	0	0	0	0	0	1	0,4456	0,3406	0,7683	0,7317
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0,4509	0,6134	0,7131
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0,4509	0,6901

TABLE 4.10. ELEMENTS OF CANONICAL EXPANSION OF ZONAL COMPONENT OF WIND VELOCITY. COLD-HALF-YEAR, MIDDLE LATITUDES

i	$\overset{\circ}{\varphi}_v H$												w_{vi}	D^*_{vi}
	0,35	0,40	0,20	0,15	0,10	0,12	0,25	0,34	0,47	0,50	0,38	0,42		
	H km													
	5	10	15	20	25	30	35	40	45	50	55	60		
1	1	0,7600	0,6700	0,4100	0,1500	0,0200	−0,0100	−0,1200	−0,0200	0,0000	0,0400	0,1000	0,3500	1,0000
2	0	1	0,5701	0,2803	0,1562	0,1771	0,1600	0,0265	0,1071	0,1420	0,1648	0,2225	0,1340	0,4224
3	0	0	1	0,4055	0,0287	−0,0146	−0,1494	−0,1595	−0,1507	−0,1068	−0,0640	−0,0014	−0,1086	0,4138
4	0	0	0	1	0,4314	0,4014	0,2056	0,0996	0,1380	0,0836	0,1298	0,0861	0,0141	0,7307
5	0	0	0	0	1	0,7329	0,5390	0,3571	0,2217	0,1752	0,0758	0,0279	0,0239	0,8309
6	0	0	0	0	0	1	0,9444	1,0723	1,0800	1,0600	1,1351	1,0400	0,0649	0,4222
7	0	0	0	0	0	0	1	0,5277	0,5517	0,4536	0,3603	0,2893	0,1722	0,3309
8	0	0	0	0	0	0	0	1	0,7232	0,5783	0,4225	0,6111	0,1912	0,2840
9	0	0	0	0	0	0	0	0	1	1,0655	0,9113	0,6644	0,1384	0,1890
10	0	0	0	0	0	0	0	0	0	1	0,8002	0,8091	0,1248	0,1043
11	0	0	0	0	0	0	0	0	0	0	1	0,8654	−0,1131	0,1068
12	0	0	0	0	0	0	0	0	0	0	0	1	−0,1657	0,1410

TABLE 4.11. ELEMENTS OF CANONICAL EXPANSIONS OF MERIDIONAL WIND VELOCITY COMPONENTS. COLD HALF-YEAR, MIDDLE LATITUDES

i	$\tilde{\varphi}_u(H)$												w_{ul}	D^*_{ul}
	0,35	0,40	0,20	0,15	0,10	0,12	0,25	0,34	0,47	0,50	0,38	0,42		
	H, km													
	5	10	15	20	25	30	35	40	45	50	55	60		
1	1	0,7200	0,5400	0,2200	0,0300	−0,0050	0,0200	0,0600	0,1700	0,2100	0,1600	−0,0100	0,3500	1,0000
2	0	1	0,7292	0,1487	−0,0864	−0,1329	−0,0922	−0,0066	−0,0257	−0,0025	0,0930	0,0149	0,1580	0,4816
3	0	0	1	0,3290	0,0091	−0,1024	−0,1070	−0,2212	−0,1609	−0,2046	−0,1527	0,0224	−0,1043	0,4522
4	0	0	0	1	0,2895	0,1410	0,1213	0,0339	−0,0130	−0,062	−0,0214	−0,0585	0,0819	0,8919
5	0	0	0	0	1	0,5430	0,4178	0,2499	0,1062	0,0766	0,0382	0,1585	0,0805	0,9207
6	0	0	0	0	0	1	0,6705	0,5368	0,4142	0,3459	0,2452	0,0714	0,0883	0,6950
7	0	0	0	0	0	0	1	0,6107	0,4070	0,2815	0,2318	0,2659	0,1654	0,5040
8	0	0	0	0	0	0	0	1	0,6441	0,5223	0,4634	0,1511	0,1268	0,5274
9	0	0	0	0	0	0	0	0	1	0,7523	0,5565	0,5900	0,2039	0,5260
10	0	0	0	0	0	0	0	0	0	1	0,6844	0,6045	0,1087	0,3629
11	0	0	0	0	0	0	0	0	0	0	1	0,8189	−0,0131	0,4425
12	0	0	0	0	0	0	0	0	0	0	0	1	0,1506	0,3093

Above $\xi(H)$ denoted $\rho(H)$, t(H), u(H), and ν(H).

In this case, to calculate the elements of the canonical expansion of the air density, air temperature, and wind velocity components, as follows from Eqs. (4.92), it is not the covariance, but the corresponding correlation matrices, and the actual canonical expansions of the listed atmospheric parameters that are to be described by the equalities:

$$\rho(H) = m_\rho(H) + \sigma_\rho(H) \sum_{i=1}^{n} w_{\rho i} x_{\rho i}(H); \tag{4.93}$$

$$t(H) = m_t(H) + \sigma_t(H) \sum_{i=1}^{n} w_{ti} x_{ti}(H); \tag{4.94}$$

$$v(H) = m_v(H) + \sigma_v(H) \sum_{i=1}^{n} w_{vi} x_{vi}(H); \tag{4.95}$$

$$u(H) = m_u(H) + \sigma_u(H) \sum_{i=1}^{n} w_{ui} x_{ui}(H). \tag{4.96}$$

As an example, below we present the values of the elements of the canonical expansions of air density and temperature, and of wind velocity components, as well as the dispersions of the random coefficients of the canonical expansions for the mid-latitude cold half-year.

Tables 4.8 - 4.11 contain the values of the centered normalized functions of the physical parameters of the atmosphere, the random coefficients of the canonical expansions, the dispersions of the random coefficients, as well as the matrices of the coordinate functions. In accordance with Eqs. (4.85) and (4.86), the matrices of the coordinate functions are triangular with unit elements along the principal diagonals. The first rows of these matrices are equal, as follows from the last equality of the system (4.92), to the first rows of the corresponding correlation matrices.

If we assume the stationary approximation of the functions of the physical parameters of the atmosphere $\phi_\xi(H)$, to describe the atmospheric perturbations we can use the spectral canonical expansions. /129

An expansion of the form /49/

$$R_\varphi(\Delta H) = \sum_{\nu=1}^{\infty} D_\nu \cos \Omega_\nu \Delta H. \tag{4.97}$$

is the spectral canonical expansion of the covariance function of a stationary random function ϕ_ξ H. To this canonical expansion of the covariance function there corresponds the canonical expansion of the random function

$$\varphi(H) = \sum_{\nu=1}^{\infty} \alpha_\nu \cos \Omega_\nu H + \beta_\nu \sin \Omega_\nu H, \tag{4.98}$$

where sin Ω_νH and cos Ω_νH are the coordinate functions, and α_ν and β_ν are the random coefficients.

The spectral canonical expansion of the random function has several features. It has unified coordinate functions and, therefore, is wholly determinate if the random coefficients, on which as before the following conditions are imposed, are found: they must be uncorrelated and, in addition, must be such that

$$M[\alpha_\nu] = M[\beta_\nu] = 0; \quad D[\alpha_\nu] = D[\beta_\nu] = D_\nu. \tag{4.99}$$

To determine the dispersions of the random coefficients of the spectral canonical expansions (4.98), let us express the dispersion of the random function ϕ(H) by its spectral density $S_\phi(\Omega)$

$$D_\varphi = 2\int_0^\infty S_\varphi(\Omega)\, d\Omega. \tag{4.100}$$

and let us divide the interval of integration into a series of elementary intervals $(\Omega_{\nu-1}, \Omega_{\nu+1})$. We will have

$$D_\varphi = \sum_{\mu=1}^{\infty} D_\mu. \tag{4.101}$$

In Eq. (4.101), we introduce the notation

$$D_\mu = 2\int_{\Omega_{\mu-1}}^{\Omega_{\mu+1}} S_\varphi(\Omega)\, d\Omega. \tag{4.102}$$

Let us substitute the value of D_μ from Eq. (4.102) in place of D_ν in Eq. (4.97), remembering that $R_\varphi(\Delta H) = r_\xi(\Delta H)$ /130

$$r_\xi(\Delta H) = \sum_{\nu=1}^{\infty} D_\nu \cos \Omega_\nu \Delta H =$$
$$= \sum_{\nu=1}^{\infty} 2\cos \Omega_\nu \Delta H \int_{\Omega_{\nu-1}}^{\Omega_{\nu+1}} S_\varphi(\Omega)\, d\Omega. \tag{4.103}$$

The function $S_\varphi(\Omega)$ in the interval $(\Omega_{\nu-1}, \Omega_{\nu+1})$ does not change its sign, and the value of Ω_ν lies within the indicated interval. Therefore, we can apply the theorem on the mean to Eq. (4.103). We will have

$$r_\xi(\Delta H) = 2\sum_{\nu=1}^{\infty} \int_{\Omega_{\nu-1}}^{\Omega_{\nu+1}} S_\varphi(\Omega)\cos\Omega\Delta H d\Omega$$

or

$$r_\xi(\Delta H) = 2\int_0^\infty S_\varphi(\Omega)\cos\Omega\Delta H d\Omega. \tag{4.104}$$

Eq. (4.104) is the Fourier transform of the spectral density of the random function ϕ(H). Therefore, the dispersions of the random coefficients of the spectral canonical expansion (4.98) are terms of the dispersions of the random function ϕ(H) and are determined by the equality

$$D_\nu = 2\int_{\Omega_{\nu-1}}^{\Omega_{\nu+1}} S_\varphi(\Omega)\, d\Omega. \tag{4.105}$$

They can be computed if we know the analytic expression for the spectral density of the random function ϕ(H).

Now let us consider the variables

$$\alpha_\nu = \sigma_\nu\gamma_\nu; \quad \beta_\nu = \sigma_\nu\varepsilon_\nu \quad (\nu = 1, 2, \ldots), \tag{4.106}$$

in which $\sigma_\nu = \sqrt{D_\nu}$ is the root mean square deviation of the random coefficients of the spectral canonical expansion, and γ and ϵ are the sets of normally distributed random numbers with zero mathematical expectation and unit dispersion. A sequence of these numbers is set up on the basis of the random numbers [9] uniformly distributed in the interval [0, 1]. Obviously, the variables α_ν and β_ν are random. Let us find the mathematical expectation and the dispersion of these random variables. Based on the properties of a mathematical expectation, we have:

$$M[\alpha_\nu] = M[\beta_\nu] = \sigma_\nu M[\gamma_\nu] = \sigma_\nu M[\epsilon_\nu] = 0;$$
$$D[\alpha_\nu] = D[\beta_\nu] = \sigma_\nu^2 M[\gamma_\nu^2] = \sigma_\nu^2 M[\epsilon_\nu^2] = D_\nu.$$

Thus, the random variable α_ν formulated from the root mean square deviations of the coefficients of the canonical expansion (4.98) of random function and of normally distributed random numbers exhibit the properties of coefficients of the spectral canonical expansion and can be easily obtained. /131

Based on the foregoing, we can write the spectral canonical expansions of the physical parameters of the atmosphere as follows:

$$\rho(H) = m_\rho(H) + \sigma_\rho(H) \sum_{\nu=1}^{n} \sigma_{\nu\rho} (\gamma_\nu \cos \Omega_\nu H + \varepsilon_\nu \sin \Omega_\nu H), \tag{4.107}$$

$$t(H) = m_t(H) + \sigma_t(H) \sum_{\nu=1}^{n} \sigma_{\nu t} (\gamma_\nu \cos \Omega_\nu H + \varepsilon_\nu \sin \Omega_\nu H), \tag{4.108}$$

$$v(H) = m_v(H) + \sigma_v(H) \sum_{\nu=1}^{n} \sigma_{\nu v} (\gamma_\nu \cos \Omega_\nu H + \varepsilon_\nu \sin \Omega_\nu H), \tag{4.109}$$

$$u(H) = m_u(H) + \sigma_u(H) \sum_{\nu=1}^{n} \sigma_{\nu u} (\gamma_\nu \cos \Omega_\nu H + \varepsilon_\nu \sin \Omega_\nu H). \tag{4.110}$$

A comparison of Eqs. (4.105) and (4.65) shows that the dispersions of the random coefficients of the spectral canonical expansions of these physical parameters of the atmosphere represent probabilities that the frequencies of the atmospheric perturbations lie within certain intervals and, therefore, take into account the features of the vertical structure of the fields of thermodynamic characteristics of the atmosphere. The dispersions of the random coefficients of air temperature and density are given in Table 4.12.

TABLE 4.12. DISPERSIONS OF RANDOM COEFFICIENTS OF CANONICAL EXPANSIONS OF AIR TEMPERATURE AND DENSITY

Group of functions	ν										
	1	2	3	4	5	6	7	8	9	10	11
	Ω_ν km^{-1}										
	0,09	0,18	0,27	0,36	0,45	0,54	0,63	0,72	0,81	0,90	0,99
I	0,060	0,203	0,527	0,101	0,039	0,021	0,016	0,007	0,006	0,005	0,005
II	0,113	0,241	0,266	0,155	0,066	0,040	0,020	0,014	0,010	0,008	0,007
III	0,332	0,213	0,117	0,088	0,040	0,029	0,028	0,025	0,012	0,008	0,008
IV	0,476	0,209	0,096	0,046	0,038	0,030	0,025	0,018	0,011	0,06	0,005

If we select as the convergence criterion the criterion

$$1 - \sum_\nu D_\nu \leqslant 0.1, \tag{4.111}$$

which derives from the confidence intervals and the precision of the approximation of the correlation functions of air temperature and density, then the largest number of terms occurs for the expansion corresponding to group III of correlation functions, and the smallest number of terms corresponds to the expansion that is part of group I of these functions, where they total five in the latter case. /132

Dispersions of random functions of canonical expansions of wind velocity components are given in Table 4.13. These data indicate that (based on the above-indicated criterion) the fastest convergence is observed for group IV, and the smallest -- for group II.

TABLE 4.13. DISPERSIONS OF RANDOM COEFFICIENTS OF CANONICAL EXPANSIONS OF WIND VELOCITY COMPONENTS

Group of functions	ν										
	1	2	3	4	5	6	7	8	9	10	11
	Ω_ν, km^{-1}										
	0,09	0,18	0,27	0,36	0,45	0,54	0,63	0,72	0,81	0,90	0,99
I	0,3381	0,2941	0,1291	0,0627	0,0372	0,0246	0,0182	0,0135	0,0111	0,0085	0,0061
II	0,2775	0,2040	0,1455	0,0881	0,0577	0,0342	0,0310	0,0214	0,0198	0,0139	0,0108
III	0,05201	0,3000	0,0463	0,0344	0,0132	0,0094	0,0064	0,0023	0,0018	0,0015	0,0013
IV	0,8152	0,0851	0,0333	0,0212	0,0145	0,0086	0,0069	0,0058	0,0043	0,0038	0,0022

4.4. Shaping Filters of Physical Parameters of the Atmosphere

One of the forms of the representations of random atmospheric perturbations can be shaping filters. Filters that permit shaping a random process whose correlation function is known, from white noise, are called shaping filters.

The spectral density $S_\varphi(\Omega)$ of a stationary random function $\phi(H)$ (as was shown in Section 4.1, centered and normalized values of air temperature and density and wind velocity components as functions of altitude can be placed in the class of stationary random functions) can be represented as a rational-fractional function Ω and can be written in the form of the product of two cofactors:

$$S_\varphi(\Omega) = S_1(\Omega) G_1(\Omega). \qquad (4.112)$$

Cofactor $S_1(\Omega)$ contains zeros and the poles of the function $S_\varphi(\Omega)$, lying in the upper half-plane and is a bounded and analytic function in the lower half-plane. Conversely, cofactor $G_1(\Omega)$ contains zeros and poles of the function $S_\varphi(\Omega)$, located in the lower half-plane and is a bounded and analytic function in the upper half-plane. For real values of Ω /133

$$G_1(\Omega) = \overline{S_1(\Omega)},$$

where the overbar denotes a complexly-conjugate variable and, thus,

$$S_\varphi(\Omega) = S_1(\Omega) \overline{S_1(\Omega)} = |S_1(\Omega)|^2. \qquad (4.113)$$

Therefore, the function $S_1(\Omega)$ exhibits all the properties of a frequency transfer function of a stable linear stationary minimum-phase system.

If white noise exhibiting, as we know, a constant spectral density and the correlation function

$$R_m(h) = \delta(h),$$

where $\delta(h)$ is the delta-function, is passed through a filter that has a frequency transfer function $\Phi(i\Omega)$, the spectral density of the output signal obviously will be

$$S_x(\Omega) = |\Phi(i\Omega)|^2. \tag{4.114}$$

Comparing Eqs. (4.113) and (4.114) we can easily show that a random process can be obtained from white noise if the latter is passed through a shaping filter whose frequency transfer function is defined by the expression

$$\Phi(i\Omega) = S_1(\Omega). \tag{4.115}$$

If the white noise has a single intensity, the spectral density of the white noise is $1/2\pi$. In this case the spectral density of the output signal differs from Eq. (4.114) by the cofactor $1/2\pi$, that is,

$$S_x(\Omega) = \frac{1}{2\pi}|\Phi(i\Omega)|^2 \tag{4.116}$$

or

$$S_x(\Omega) = \frac{1}{2\pi}S_1(\Omega)\overline{S_1(\Omega)} = \frac{1}{2\pi}\frac{H(i\Omega)}{F(i\Omega)}\frac{H(-i\Omega)}{F(-i\Omega)},$$

where $H(i\Omega)$ and $F(i\Omega)$ are polynomials in Ω, in which all zeros are in the upper half-plane symmetrically with respect to the imaginary semiaxis. In addition, the zeros of each of the polynomials $H(i\Omega)$ and $F(i\Omega)$ are pairwise conjugate complex numbers lying in the left half-plane of the variable $i\Omega$. Therefore, all coefficients of the polynomials $H(i\Omega)$ and $F(i\Omega)$ are positive, the frequency transfer function of the shaping filter is /134

$$\Phi(i\Omega) = \frac{H(i\Omega)}{F(i\Omega)}, \tag{4.117}$$

and the stationary random function $\varphi_\xi(h)$ is associated with the white noise v by the linear differential equation

$$F(s)\varphi_\xi = H(s)\tilde{v}. \tag{4.118}$$

In Eqs. (4.118), F(s) and H(s) are polynomials in the differentiation operator $s = d/dh$ with constant coefficients

$$\left.\begin{aligned}F(s) &= a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0,\\ H(s) &= b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0.\end{aligned}\right\} \quad (4.119)$$

Now let us examine the shaping filters linking the physical parameters of the atmosphere with white noise.

In Section 4.1 it was shown that centered and normalized values of air temperature and density and wind velocity components, being functions of altitude, can be placed in the class of stationary random functions, and depending on the atmospheric layer, their correlation functions are determined by the equality

$$r_\beta(\Delta h) = e^{-k\Delta h} \quad (4.120)$$

or by the expression

$$r_\xi(\Delta h) = e^{-k\Delta h}\cos\Omega_0\Delta h. \quad (4.121)$$

If the correlation function of the above-listed random functions is expressed by the exponential (4.120), the normalized spectral density is

$$S_\varphi(\Omega) = \frac{1}{\pi}\frac{k}{k^2+\Omega^2}. \quad (4.122)$$

Obviously, Eq. (4.122) is equivalent to the expression

$$S_\varphi(\Omega) = \frac{1}{2\pi}\left|\frac{\sqrt{2k}}{k+i\Omega}\right|^2. \quad (4.123)$$

By comparing Eqs. (4.123) and (4.116), we see that the random function ϕ can be viewed as the result of the indefinitely long passage of unit-intensity white noise through a shaping filter having the frequency transfer function

$$\Phi(i\Omega) = \frac{\sqrt{2k}}{k+i\Omega}. \quad (4.124)$$

Now let us compare the frequency transfer functions (4.124) and (4.117). Obviously, the polynomial $H(i\Omega)$ is of zero degree, and the polynomial $F(i\Omega)$ is of the first degree. Thus, Eq. (4.118) becomes /135

$$a_1\frac{d\varphi(h)}{dh} + a_0\varphi(h) = b_0\tilde{v}, \quad (4.125)$$

where

$$a_0 = k;\ a_1 = 1;\ b_0 = \sqrt{2k}.$$

For the cases when the correlation function of a random function is described by Eq. (4.121) (groups I and II are correlation functions of air temperature and density), the spectral density is of the form

$$S_\varphi(\Omega) = \frac{k}{\pi}\,\frac{\beta^2+\Omega^2}{\beta^4+2\left(k^2-\Omega_0^2\right)\Omega^2+\Omega^4}\ (\beta^2=\Omega^2+k^2). \qquad (4.126)$$

The numerator of Eq. (4.126) has two purely imaginary roots $\Omega = \pm i\beta$. But the denominator has four complex groups $\Omega = \pm\Omega_0 \pm ik$, which lie symmetric relative to the real and imaginary axes. By selecting the roots lying on the upper half-plane, that is, $\Omega = i\beta$ and $\Omega = \pm\Omega_0 + ik$, let us write out Eq. (4.126) for the real values of Ω, which will be of the form:

$$S_\varphi(\Omega) = \frac{1}{2\pi}\left|\frac{\sqrt{2k}\,(\beta+i\Omega)}{\beta^2+2ki\Omega+(i\Omega)^2}\right|^2. \qquad (4.127)$$

Like (4.123), Eq. (4.127) shows that the random function $\phi(h)$ can be regarded as resulting from the passage of white noise through a stationary linear system (shaping filter) whose frequency transfer function is expressed by the equality

$$\Phi(i\Omega) = \alpha\,\frac{\beta+i\Omega}{(i\Omega)^2+2ki\Omega+\beta^2} \quad \text{when } \alpha = \sqrt{2k}. \qquad (4.128)$$

Since the numerator in Eq. (4.128) is a polynomial of the first degree, and the denominator is a polynomial of the second degree, the equation linking the random function $\phi(h)$ with white noise $\tilde{v}$ is of the form

$$a_2\frac{d^2\varphi(h)}{dh^2} + a_1\frac{d\varphi(h)}{dh} + a_0\varphi(h) = b_1\frac{d\tilde{v}}{dh} + b_0\tilde{v}. \qquad (4.129)$$

in which

$$a_0 = \beta^2;\ \ a_1 = 2k;\ \ a_2 = 1;\ \ b_0 = \beta\sqrt{2k};\ \ b_1 = \sqrt{2k}.$$

Table 4.14 gives the values of the coefficients of differential equations (4.125) and (4.129) for different groups of correlation functions of air temperature and density (the number of the function group, as indicated in Section 4.1, corresponds to a specific atmospheric layer, different latitudes, and different half-year periods). /136

TABLE 4.14. COEFFICIENTS OF DIFFERENTIAL EQUATIONS LINKING FLUCTUATIONS IN AIR DENSITY AND TEMPERATURE WITH WHITE NOISE

Coefficient	Group of functions I	II	III	IV
a_0	0.0475	0.0529	0.1570	0.0975
a_1	0.0800	0.1880	1.0000	1.0000
a_2	1.0000	1.0000	—	—
b_0	0.0620	0.1005	0.5648	0.4425
b_1	0.2840	0.4350	—	—

TABLE 4.15. COEFFICIENTS OF DIFFERENTIAL EQUATIONS LINKING FLUCTUATIONS IN WIND VELOCITY COMPONENTS WITH WHITE NOISE

Coefficient	Group of functions I	II	III	IV
a_0	0.0164	0.1900	0.0550	0.0275
a_1	0.2000	1.0000	1.0000	1.0000
a_2	1.0000	—	—	—
b_0	0.0575	0.6102	0.3321	0.2352
b_1	0.4478	—	—	—

The corresponding functions for the meridional and zonal components of wind velocity are in Table 4.15.

In Table 4.15, just as for air temperature and density, the number of the correlation function group determines the wind velocity component, the atmospheric layer, the latitudinal group, and the half-year. The placement of each of the groups correlation functions of wind velocity in the above-listed situations is determined in Section 4.1.

4.5. Representation of Atmospheric Perturbations Using Eigenelements of the Correlation Matrices of the Physical Parameters of the Atmosphere

An expansion of atmospheric perturbations in eigenelements of the correlation matrices of the physical parameters of the atmosphere, or in the principal components is highly attractive. It is shown in [4] that these expansions are optimal. Moreover, in several cases the elements of these expansions can be given a definite physical mean. Therefore, expansions in eigenelements of correlation matrices are finding growing use of late.

Let us represent the normalized centered function referring to some physical parameter of the atmosphere $\xi(t, H)$ in the form of the series

$$\varphi_\xi(t, H) = \sum_\nu z_\nu(t)\,\psi_\nu(H). \qquad (4.130)$$

Eq. (4.130) is the discrete representation of the function $\phi(t, H)$ /137
(the subscript ξ will be omitted in the following treatment). Therefore we can introduce the notation:

$$\begin{aligned} \varphi(t, H) &= \varphi_{ij} \quad (i = 1, 2, \ldots, m), \\ u_\nu(H) &= u_{\nu j} \quad (j = 1, 2, \ldots, n), \\ z_\nu(t) &= z_{\nu i} \quad (\nu = 1, 2, \ldots). \end{aligned}$$

We will seek Eq. (4.130) in the class of the optimal approximations to the function ϕ_{ij} in the sense of least squares. To do this, let us find the minimum of the expression

$$\Delta = \sum_i \sum_j \left[\varphi_{ij} - \sum_\nu z_{\nu i} u_{\nu j} \right]^2. \tag{4.131}$$

Let us assume that the functions $z_{\nu i}$ and $u_{\nu j}$ are orthogonal, that is,

$$\sum_i z_{\nu i} z_{\mu i} = 0 \quad \text{for} \quad \nu \neq \mu,$$
$$\sum_j u_{\nu j} u_{\mu j} = 0 \quad \text{for} \quad \nu \neq \mu.$$

Then Eq. (4.131) can be rewritten thusly:

$$\Delta = \sum_i \sum_j \left[\varphi_{ij}^2 - 2\varphi_{ij} \sum_\nu z_{\nu i} u_{\nu j} + \sum_\nu z_{\nu i}^2 u_{\nu j}^2 \right]. \tag{4.132}$$

To determine the minimum of Eq. (4.132), it is necessary to set equal to zero the derivatives $\frac{\partial \Delta}{\partial z_{\nu i}}$ and $\frac{\partial \Delta}{\partial u_{\nu j}}$. As a result, we get these expressions

$$\sum_i \varphi_{ij} z_{\nu i} = u_{\nu j} \sum_i z_{\nu i}^2, \tag{4.133}$$

$$\sum_j \varphi_{ij} u_{\nu j} = z_{\nu i} \sum_j u_{\nu j}^2. \tag{4.134}$$

Suppose the functions $u_{\nu j}$ are normalized, that is,

$$\sum_j u_{\nu j}^2 = 1. \tag{4.135}$$

Then

$$z_{\nu i} = \sum_j \varphi_{ij} u_{\nu j}. \tag{4.136}$$

Substituting Eq. (4.136) into Eq. (4.133), and using the notation /138

$$\lambda_\nu = \frac{1}{m} \sum z_{\nu i}^2 \tag{4.137}$$

and

$$r_{jk} = \frac{1}{m} \sum \varphi_{ij} \varphi_{ik}, \tag{4.138}$$

we arrive at the equality

$$\sum_k r_{jk} u_{\nu k} = \lambda_\nu u_{\nu j} \quad (j = 1, 2, \ldots, n), \tag{4.139}$$

which in matrical expression is of the form

$$ru = \lambda u. \tag{4.140}$$

Obviously, in Eq. (4.140) λ are eigenvalues, and u are the corresponding eigenvectors of the correlation matrix r. Eq. (4.140) means that we are seeking for such a vector u that is transformed by the matrix r into vectors differing from the initial vectors by the scalar multiplier λ. The matrical equation (4.140) in the expanded form can be represented thusly:

$$\left. \begin{array}{l} (r_{11} - \lambda) u_1 + r_{12} u_2 + \ldots + r_{1n} u_n = 0 \\ r_{21} u_1 + (r_{22} - \lambda) u_2 + \ldots + r_{2n} u_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ r_{n1} u_1 + r_{n2} u_2 + \ldots + (r_{nn} - \lambda) u_n = 0 \end{array} \right\}. \tag{4.141}$$

These equations as we can see comprise a system of linear equations in u_i. A nontrivial solution to this system exists if and only if λ are the roots of the characteristic equation

$$|r - \lambda E| = 0. \tag{4.142}$$

Correlation matrices of the physical parameters of the atmosphere are real symmetric matrices. Therefore the eigenvalues are also real. In addition, to satisfy Eq. (4.137), they must be positive.

Using the orthogonal transformation, correlation matrix r can be reduced to the diagonal form. If T is the orthogonal matrix of eigenvectors, and T^* is the transposed matrix corresponding to it, then

$$T^* r T = \Lambda, \tag{4.143}$$

where Λ is the diagonal matrix with elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ in decreasing order. This requirement on the arrangement of λ_ν (ν = 1, 2, ..., n) is customary and achieves uniqueness. If we take the determinant of the matrix in Eq. (4.143), we get /139

$$|\Lambda| = |T^*||r||T| = |r| \tag{4.144}$$

and, therefore,

$$|r| = \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n. \tag{4.145}$$

Eq. (4.145) shows that if we know the eigenvalues of the correlation matrix, it is not difficult to compute the determinant of the latter. Multiplying Eq. (4.143) at the right by T* and at the left by T, we get

$$r = T\Lambda T^*. \tag{4.146}$$

The matrix that is the reciprocal of matrix r is

$$r^{-1} = T\Lambda^{-1}T^*. \tag{4.147}$$

The latter equality can be used in finding a matrix that is the reciprocal of the correlation matrix when we know the eigenvalues and the eigenvectors of the correlation matrix are known. In addition, from Eq. (4.143) we can obtain an important property of correlation matrices. To do this, let us take the trace of the matrix Λ in Eq. (4.143)

$$S_p\Lambda = S_p(T^*rT) = S_p(rTT^*) = S_p(rE) = S_pr. \tag{4.148}$$

Eq. (4.148) indicates that the sum of the elements along the principal diagonal of the correlation matrix is equal to the sum of its eigenvalues.

As indicated above, the eigenvalues are the roots of the characteristic equation (4.142), which is an equation of degree n in λ. After they have been computed, we can solve the system of equations (4.141) for determining the eigenvectors of matrix r. Actually, by assuming the system (4.141) successively $\lambda=\lambda_1$, $\lambda=\lambda_2$, and so on we get n different solutions to system (4.141), which then will represent eigenvectors of the correlation matrix r:

$$\begin{array}{ll} u_1;\ u_{11};\ u_{12};\ldots;\ u_{1n} & (\text{for } \lambda = \lambda_1) \\ u_2;\ u_{21};\ u_{22};\ldots;\ u_{2n} & (\text{for } \lambda = \lambda_2) \\ & \ldots\ldots\ldots \\ u_n;\ u_{n1};\ u_{n2};\ldots;\ u_{nn} & (\text{for } \lambda = \lambda_n). \end{array}$$

TABLE 4.16. EIGENVALUES λ_ν AND THE EIGENVECTORS OF THE CORRELATION MATRIX OF AIR DENSITY CORRESPONDING TO THEM. MIDDLE LATITUDES, COLD HALD-YEAR

H km	ν 1	2	3	4	5	6	7	8	9	10	11	12	13	14
λ_ν	3,1028	2,4611	2,0914	1,3153	0,9993	0,7812	0,6846	0,6537	0,5205	0,4617	0,3204	0,3091	0,2270	0,0717
3	–0,1239	0,4076	–0,2604	0,3382	0,0373	–0,3581	0,1276	–0,4011	0,0225	0,0858	–0,1320	–0,0354	–0,1075	0,5397
6	–0,1202	0,3130	–0,1955	0,3367	0,2326	0,4104	–0,0081	0,6331	–0,1189	0,2068	0,1571	–0,0056	0,0250	0,1744
9	0,0438	–0,1587	0,1031	0,1486	0,9132	–0,1347	0,1166	–0,1199	0,1306	–0,0332	–0,0003	–0,0921	–0,0234	–0,1815
15	0,2253	–0,4991	0,2193	0,0135	0,0975	0,0997	–0,2139	0,0185	–0,1751	–0,0896	0,1609	0,2344	–0,0935	0,6768
20	0,1138	–0,4532	–0,0366	0,2564	–0,1656	0,1199	0,4082	–0,0184	0,1459	0,5853	–0,2968	0,1825	0,1379	0,0385
25	0,0341	–0,2997	–0,2151	0,4086	–0,1680	–0,5485	0,1736	0,3886	–0,1508	–0,3537	0,1228	–0,0177	–0,0368	–0,1456
30	0,0860	–0,1846	–0,4890	0,2493	–0,0139	0,1237	–0,6676	–0,0680	0,3235	–0,0179	–0,2567	0,0184	–0,2203	–0,1345
35	0,2351	–0,0743	–0,4979	0,0413	0,0323	0,1780	0,0461	–0,3951	–0,3264	0,0830	0,4811	–0,1475	0,3195	–0,1728
40	0,3191	0,0169	–0,3552	–0,2447	0,0575	0,3035	0,3652	0,0349	–0,0712	–0,4595	–0,2931	0,3311	–0,2076	0,1641
45	0,3904	0,1197	–0,1893	–0,2757	0,0058	–0,1703	0,1713	0,2375	0,4665	0,1070	–0,0274	–0,6068	0,0651	0,0030
50	0,4400	0,1340	0,0032	–0,1824	–0,0188	–0,3116	–0,1779	0,1245	0,0734	0,3268	0,3816	0,5557	–0,1930	0,0812
55	0,4252	0,1990	0,1396	0,0798	0,0708	–0,1559	–0,2618	0,0863	–0,3986	0,0120	–0,4953	0,0132	0,4932	–0,0250
60	0,3770	0,1754	0,2762	0,3195	–0,0872	0,1477	0,1190	–0,1306	–0,2422	0,0929	–0,0245	–0,2678	–0,6095	–0,2814
65	0,2644	0,1543	0,2986	0,4219	–0,1565	0,2205	0,0509	–0,1176	0,4866	–0,3572	0,2307	0,1456	0,3290	0,0496

TABLE 4.17. EIGENVALUES λ_ν AND THE EIGENVECTORS OF THE CORRELATION MATRIX OF AIR TEMPERATURE CORRESPONDING TO THEM. MIDDLE LATITUDES, COLD PERIOD

H km	ν													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	λ_ν													
	3,4721	3,0541	1,9542	1,1174	0,9327	0,6365	0,5274	0,4702	0,4268	0,3969	0,3307	0,2914	0,2174	0,1720
3	-0,0364	0,4598	0,0723	0,2467	0,2311	-0,0801	0,2446	0,1986	0,1225	-0,2695	-0,2718	0,5297	-0,3420	0,0544
6	-0,0721	0,5076	0,1049	-0,0331	0,1441	0,0282	0,0220	0,0015	0,0708	0,2739	-0,0928	0,0442	0,5048	-0,5980
9	-0,1001	0,5034	0,0729	0,0119	0,0265	0,1849	-0,1580	-0,0053	0,0857	0,2407	0,2743	-0,1078	0,1115	0,7098
15	0,1564	-0,4001	-0,1182	0,4352	0,0132	0,0807	0,0521	0,0808	0,2981	0,2783	0,0455	0,4731	0,4195	0,1647
20	0,1441	0,1316	-0,1543	0,8093	0,0687	-0,1388	-0,0756	-0,0873	-0,1524	-0,0517	0,0784	-0,4408	-0,0889	-0,1078
25	0,2840	0,2215	-0,3139	-0,0699	-0,3678	-0,2360	-0,1129	-0,2393	-0,4829	-0,2423	-0,0480	0,3222	0,3401	0,1080
30	0,3152	0,1734	-0,3204	-0,0968	-0,4070	-0,1431	-0,0174	0,0147	0,3472	0,4227	0,1771	0,0658	-0,4575	-0,1662
35	0,3829	0,1184	-0,2807	-0,1806	-0,0266	0,0469	0,2281	0,2733	0,3894	-0,2803	-0,3314	-0,3937	0,3063	0,1276
40	0,3533	0,0291	-0,1688	-0,0180	0,1987	0,7972	-0,2416	0,0328	-0,2305	0,0005	-0,0396	0,1049	-0,1652	-0,1348
45	0,4018	-0,0637	0,0701	-0,1444	0,3712	-0,1995	0,3173	0,2359	0,2012	-0,0832	0,6516	0,0341	-0,0226	0,0328
50	0,3772	-0,0238	0,1919	-0,0994	0,3992	-0,1904	0,1207	-0,5410	-0,0515	0,3740	-0,3944	-0,0367	-0,0626	0,1098
55	0,3086	-0,0046	0,3913	-0,0107	0,0519	-0,1813	0,6816	-0,0407	0,3556	-0,3224	0,0948	0,0768	0,0171	-0,0639
60	0,2387	0,0174	0,4752	0,0749	-0,3047	-0,0503	-0,0357	0,5691	-0,3363	0,3068	-0,2738	-0,0704	0,0194	0,0588
65	0,1755	0,0339	0,4599	0,1260	0,4407	0,3276	0,4421	-0,3695	0,1390	-0,2316	0,1574	-0,0214	0,0131	-0,0495

TABLE 4.18. EIGENVALUES λ_ν AND THE EIGENVECTORS OF THE CORRELATION MATRIX OF THE MERIDIONAL WIND VELOCITY COMPONENTS. COLD HALF-YEAR, MIDDLE LATITUDES

H km	ν											
	1	2	3	4	5	6	7	8	9	10	11	12
	λ_ν											
	4,0030	2,5176	1,8315	0,9775	0,6990	0,4997	0,3990	0,3798	0,2673	0,1920	0,1741	0,1193
5	0,1065	0,5040	−0,0816	−0,1936	−0,1522	−0,4931	0,1170	0,4789	0,1733	−0,3709	0,1063	0,0464
10	0,0823	0,5621	−0,0709	−0,1493	−0,1820	0,0823	−0,0489	0,0404	−0,2898	0,5645	−0,4350	0,1185
15	0,0259	0,5338	−0,1152	0,0370	−0,1104	0,5097	−0,1729	0,4006	0,1562	−0,3130	0,3226	−0,1040
20	0,0419	0,2327	−0,3524	0,4977	0,7340	−0,0180	−0,2141	−0,1498	−0,0222	0,0443	−0,0378	0,0792
25	0,1760	−0,0896	−0,4838	0,3920	−0,3841	−0,4116	0,2336	0,4196	−0,0098	0,0649	0,0650	−0,1307
30	0,2942	−0,2025	−0,4143	−0,0529	−0,1970	0,0986	−0,6571	−0,2166	−0,2853	−0,2078	−0,1518	0,1511
35	0,3452	−0,1558	−0,3397	0,1734	−0,0465	0,3645	0,2259	−0,3552	0,4589	0,3366	0,1246	−0,2437
40	0,3907	−0,1010	−0,1170	−0,3613	0,2274	0,1231	0,5023	0,1863	−0,3876	−0,2610	0,0987	0,3330
45	0,4201	0,0073	0,1448	−0,1922	0,2569	−0,1864	−0,1553	0,3500	0,3770	−0,1677	−0,5338	−0,2510
50	0,4132	0,0481	0,2611	−0,0474	0,1075	−0,2710	−0,3344	0,1381	0,0729	0,4076	0,5461	0,2665
55	0,3826	0,0687	0,3399	0,2558	−0,3105	0,0600	0,0571	−0,2048	−0,4674	−0,0583	0,0962	−0,6220
60	0,3104	−0,0136	0,3335	0,5191	−0,2719	0,2257	0,1526	−0,1368	0,2298	−0,1316	−0,2216	0,4853

TABLE 4.19. EIGENVALUES λ_ν AND EIGENVECTORS OF THE CORRELATION MATRIX OF THE ZONAL WIND VELOCITY COMPONENTS. MIDDLE LATITUDES, COLD HALF-YEAR

H km	ν = 1	2	3	4	5	6	7	8	9	10	11	12
λ_ν	5,6721	2,8954	1,4951	0,6035	0,3172	0,2833	0,2142	0,1722	0,1648	0,1024	0,0447	0,0351
5	0,0272	0,4963	—0,1711	—0,3275	—0,6539	0,0320	—0,3869	0,1815	0,0225	0,0636	—0,0168	0,0167
10	0,0572	0,5109	—0,1714	—0,2994	0,0901	0,0033	0,7012	—0,3179	0,1071	0,0511	0,0377	0,0162
15	0,0263	0,5130	—0,1460	0,0116	0,6896	—0,1462	—0,4339	0,0847	—0,1462	—0,0024	0,0150	—0,0264
20	0,0913	0,3973	0,2100	0,8036	—0,2426	—0,2549	0,1042	—0,0525	0,0246	—0,0449	—0,0029	0,0633
25	0,1809	0,1808	0,6405	—0,1410	0,0202	0,4623	—0,1005	—0,2047	—0,1410	—0,4427	0,1427	—0,0448
30	0,3404	0,0512	0,4103	—0,0404	0,1044	0,1689	—0,0094	0,1419	0,2306	0,7432	—0,2147	—0,0179
35	0,3574	—0,0291	0,2334	—0,2487	0,0167	—0,4970	0,2512	0,5863	—0,1255	—0,2803	—0,0628	0,0266
40	0,3724	—0,1425	0,0312	—0,1443	—0,0058	—0,4192	—0,2706	—0,4533	0,4442	—0,0468	0,3170	0,2543
45	0,3939	—0,0885	—0,1307	0,0004	—0,1026	—0,1662	—0,0681	—0,3543	—0,3312	0,0296	—0,2538	—0,6903
50	0,3856	—0,0747	—0,2144	0,0450	—0,0295	0,1559	—0,0011	—0,1475	—0,4988	0,0384	—0,2709	0,6535
55	0,3773	—0,0419	—0,2691	0,1596	—0,0699	0,2673	0,0907	0,2536	—0,1351	0,1353	0,7477	—0,1347
60	0,3544	0,0009	—0,3264	0,1595	0,0822	0,3496	—0,0064	0,1822	0,5506	—0,3762	—0,3608	—0,0179

Thus, the elements of expansion (4.130) proved to be determinate. Practical methods of obtaining eigenelements of correlation matrices can be found, for example, in [73].

Tables 4.16 - 4.19 contain the eigenvalues and the eigenvectors of correlation matrices of air temperature and density and of wind velocity components.

It is not difficult to see that for all of these atmospheric parameters considered, the sums of the eigenvectors are extremely close to the traces of the corresponding correlation matrices. A small difference in the fourth decimal place is determined by the specified precision of computations and by rounding-off. /144

Fig. 4.20 gives the coefficients of expansions (4.130) (principal components) computed for the profiles of air temperature, air density, and meridional and zonal wind velocity components indicated in Tables 4.8 - 4.11.

TABLE 4.20. COEFFICIENTS OF EXPANSIONS OF AIR TEMPERATURE t, AIR DENSITY ρ, MERIDIONAL u, AND ZONAL v WIND VELOCITY COMPONENTS

z	t	ρ	u	v
z_1	2,0421	0,8646	1,0335	0,9860
z_2	2,2787	—0,0392	0,4842	0,4143
z_3	1,2696	—0,7350	0,1167	—0,3479
z_4	1,3934	0,3650	—0,1347	—0,1187
z_5	0,9247	0,1616	—0,0921	—0,0319
z_6	0,0469	—0,0250	0,0265	—0,0048
z_7	0,2286	—0,2010	0,0126	0,0329
z_8	0,4349	—0,0968	0,3179	—0,1322
z_9	0,8338	0,0039	—0,0469	—0,0608
z_{10}	0,3390	0,1046	0,0425	—0,0732
z_{11}	—0,2192	0,2580	0,2871	—0,0273
z_{12}	1,1289	—0,0243	0,2579	0,0466
z_{13}	1,6052	0,0814	—	—
z_{14}	0,1189	0,0617	—	—

We can suggest another extension of the physical parameters of the atmosphere, somewhat different from Eq. (4.130), based on the eigenelements of the corresponding correlation matrices. Suppose we have a vector of initial values of parameter ϕ. Let us determine the orthogonal transformation

$$y = T^*\varphi, \tag{4.149}$$

where T as before is the orthogonal matrix, and let us find the dispersion of the variables (4.149), denoting it by Λ. Obviously,

$$\Lambda = M[yy^*] = M[T^*\varphi\varphi^*T] = T^*rT. \qquad (4.150)$$

By comparing Eqs. (4.150) and (4.143), we see that if we take in transformation (4.149) the matrix of eigenvectors as the orthogonal matrix, then the corresponding eigenvalues of the correlation matrices of initial parameters are dispersions of the new variables y_μ ($\mu = 1, 2, \ldots, n$).

In order to obtain the μ-th principal components z_μ, we will /145
normalize the variables y_μ. This means that we will "adjust them" so that the dispersions are equal to unity for $\mu = 1, 2, \ldots, n$. Obviously, this aim will be achieved if we assume

$$z = \Lambda^{-1/2}y. \qquad (4.151)$$

Referring to Eq. (4.149), we have

$$z = \Lambda^{-1/2}T^*\varphi \qquad (4.152)$$

or

$$\varphi = T\Lambda^{-1/2}z. \qquad (4.153)$$

If we introduce a notation

$$W = T\Lambda^{1/2}, \qquad (4.154)$$

ultimately we reach the extension

$$\varphi_i = \sum_{\mu=1}^{n} W_{i\mu}z_\mu, \qquad (4.155)$$

in which the principal components z_μ and their weights $W_{i\mu}$ are determined by Eqs. (4.152) and (4.154), respectively.

Using Eqs. (4.150) and (4.154), we can easily show that

$$WW^* = r \quad \text{and} \quad W^*W = \Lambda.$$

As was indicated, $S_p r = S_p\Lambda$. This means that the total dispersion of the variables ϕ_i is equal to the total dispersion of the unnormed components y_μ. Thus, we can find the fraction introduced by each component or by a series of components to the total dispersion. We denote with R^2

$$R^2 = \frac{\sum_{\mu=1}^{p} \lambda_\mu}{S_p r}. \qquad (4.156)$$

Since for the correlation matrices $S_p r = n$, where n is the order of the correlation matrix, criterion (4.156) can be rewritten as

$$R^2 = \frac{\sum_{\mu=1}^{p} \lambda_\mu}{n}. \tag{4.157}$$

Fig. 4.5 shows the variations in the criterion R^2 as a function of p. Criterion (4.157) has the significance of the relative precision of expansion (4.130) then in it a specified number p of terms is used. Using Fig. 4.5, by specifying the precision of the expansion, we can determine the number of the terms in series (4.130) needed to achieve the specified precision. In Fig. 4.5 it follows that the optimal convergence is shown by the zonal wind velocity component. /146

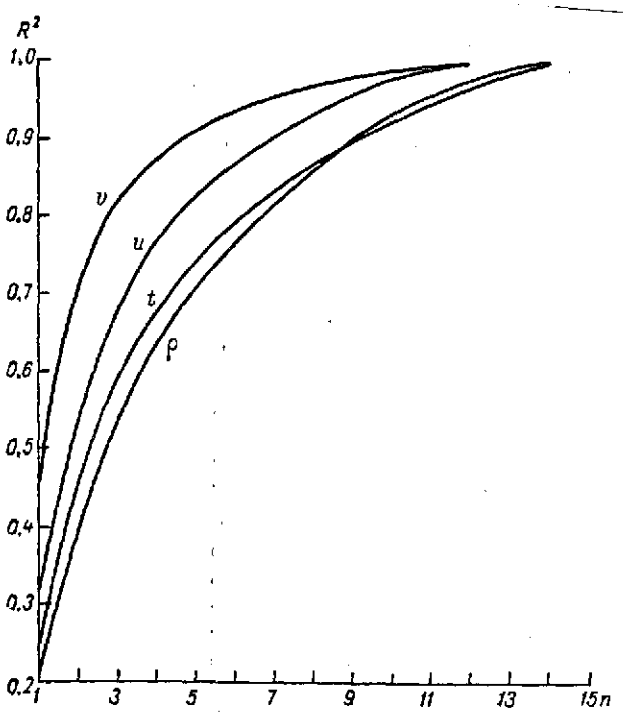


Fig. 4.5. Dependence of criterion R^2 on the number of terms in the expansion

Using the eigenelements of the correlation functions of the physical parameters of the atmosphere to determine the expansions for the normalized centered functions ϕ_ξ, by means of Eq. (4.6) we can write out the corresponding extensions for the particular physical parameters of the atmosphere we are considering:

$$\xi(t, H) = m_\xi(H) + \sigma_\xi(H) \sum_{r=1}^{p} z_r(t) u_r(H) \quad (p \leqslant n). \tag{4.158}$$

It must be noted that expansion (4.158) essentially is canonical. Actually, functions $u_r(H)$ are nonrandom and can be treated as coordinate functions, and the principal components $z_r(t)$ can be treated as uncorrelated random variables with zero mathematical expectation. The lack of correlation of the principal components and the zero equality of their mathematical expectation is determined by Eq. (4.152).

4.6. Discrete-continuous Representation of Random Processes Within the Frame of Reference of Correlation Theory /147

The above-examined canonical and noncanonical expansions of random processes are essentially representations in the class of continuous coordinate functions and continuous random variables with specified distribution laws.

Within the frame of reference of correlation theory, we can obtain expansions of random functions in the class of continuous coordinate functions and random variables of the discrete type with specified distribution. Let us show this by introducing the sequence of independent random variables of the discrete type

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_m, \qquad (4.159)$$

each element Λ_i of which in the realizations can take on the specified number of preassigned states λ_i^j $(j = 1, 2, \ldots, n)$ with the probability $P[\lambda_i^j]$. Here let us assume that the following system of identities is satisfied

$$\sum_{j=1}^{n} P[\lambda_i^j] = 1 \quad (i = 1, 2, \ldots, m). \qquad (4.160)$$

Representing the centered random process $\overset{\circ}{\xi}(t)$ in the form of the canonical expansion

$$\overset{\circ}{\xi}(t) = \sum_{\nu=1}^{m} \Lambda_\nu x_\nu(t)$$

and using the conditions

$$M[\Lambda_i] = 0 \quad (i = 1, 2, \ldots, m), \qquad (4.161)$$

$$M[\Lambda_i \Lambda_j] = 0 \quad (i \neq j = 1, 2, \ldots, m),$$

$$M[\Lambda_i^2] = \sum_{j=1}^{n} (\lambda_i^j)^2 P[\lambda_i^j] = D_i \quad (i = 1, 2, \ldots, m), \qquad (4.162)$$

we can in accordance with Eq. (4.71) write out the identities

$$R_{\xi}(t, t') = \sum_{\nu=1}^{m} D_{\nu} x_{\nu}(t) v_{\nu}(t'),$$

$$D_{\xi}(t, t) = \sum_{\nu=1}^{m} D_{\nu} [x_{\nu}(t)]^2.$$

Using the methods presented in Section 4.3, we can find the coordinate functions $x_{\nu}(t)$ and the dispersions of random variables D_{ν} ($\nu = 1, 2, \ldots, m$).

If above the process of constructing the canonical expansions is terminated at this point, then under the discrete continuous representation of the random process is necessary to set up, based on specified dispersions of the random variables of the discrete type D_{ν}, to construct their distribution series. To solve this problem, let us use Eqs. (4.161) and (4.162). Note that the problem of constructing the distribution series has a set of solutions. Let us consider a series of its solutions. /148

To satisfy condition (1.61), the number of states of random variable Λ must be greater than or equal to two. Suppose the random variable has two states λ_1 and λ_2 with probabilities P_1 and P_2. Then Eqs. (4.161) and (4.162) and condition (4.160) enables us to write out the system of equations

$$\begin{aligned} M[\Lambda] &= \lambda_1 P_1 + \lambda_2 P_2 = 0; \\ M[\Lambda^2] &= \lambda_1^2 P_1 + \lambda_2^2 P_2 = D; \\ P_1 + P_2 &= 1. \end{aligned} \qquad (4.163)$$

One equation is lacking for the solution of system (4.163), therefore let us assume that $P_1 = P_2$. Then we will have

$$\begin{aligned} P_1 = P_2 &= \frac{1}{2}; \\ \lambda_1 &= -\lambda_2; \\ \lambda = |\lambda_1| = |\lambda_2| &= \sqrt{D}. \end{aligned} \qquad (4.164)$$

Thus, using Eq. (4.164), let us present in Table 4.21 the moduli of the random variables of the discrete type for the canonical expansion of atmospheric perturbations whose characteristics are given in Tables 4.8 - 4.11 (cold half-year, middle latitudes).

TABLE 4.21. MODULI OF THE STATES OF DISCRETE-TYPE RANDOM VARIABLES

Meridional component of wind velocity	Zonal component of wind velocity	Air temperature	Air density
1,000	1,000	1,000	1,000
0,695	0,650	0,73	0,936
0,674	0,644	0,916	0,996
0,942	0,856	0,777	0,581
0,96	0,912	0,858	0,87
0,834	0,615	0,945	0,91
0,709	0,576	0,71	0,851
0,726	0,532	0,756	0,826
0,725	0,435	0,829	0,841
0,602	0,330	0,842	0,770
0,664	0,338	0,79	0,716
0,556	0,375	0,856	0,717
—	—	0,857	0,634
—	—	0,832	0,646

It is inconvenient to use the data listed in Table 4.21, therefore we can replace them with quantities that are equal to zero for all the random factors, by multiplying in advance all the coordinate functions by the magnitude of the state listed in this table. /149

Now all the absolute values of the states of the random variables of the sequence (4.159) will be equal to unity, and the states has such will be determined by the values $\lambda_1 = 1$ and $\lambda_2 = -1$.

The number of possible states of the random sequence (4.159) here is 2^m, where m is the number of discrete random variables, and the probability of each state will be determined here by the quantity 2^{-m}, since each state is equiprobable.

We can similarly perform calculations when the number of states is greater than 2. Thus, when n = 3 we will have the system of equations:

$$\left.\begin{aligned} M[\Lambda] &= \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 = 0; \\ M[\Lambda^2] &= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3 = 0; \\ P_1 + P_2 &+ P_3 = 1. \end{aligned}\right| \qquad (4.165)$$

Obviously, in this case

$$P_1 = P_2 = P_3 = \frac{1}{3};$$

$$\left.\begin{aligned} &\lambda_2 = 0;\ \lambda_1 = \lambda_3; \\ &|\lambda_1| = |\lambda_3| = \lambda = \sqrt{1.5D}. \\ &\lambda_1 = \sqrt{1.5} \approx 1.22,\ \lambda_2 = -\sqrt{1.5} \approx -1.22. \end{aligned}\right\} \qquad (4.166)$$

When D = 1, we get $\lambda_1 = \sqrt{1.5} \approx 1.22$, $\lambda_2 = -\sqrt{1.5} \approx -1.22$. The number of states when n = 3 is considerably increased and is equal to 3^m, and the probability of each of them is reduced and is equal to 3^{-m}.

When n = 4, we can easily obtain the following equations:

$$\left.\begin{aligned} M[\Lambda] &= \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4 = 0; \\ M[\Lambda^2] &= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3 + \lambda_4^2 P_4 = D; \\ P_1 &+ P_2 + P_3 + P_4 = 1, \end{aligned}\right\} \qquad (4.167)$$

whence it follows that

$$P = P_1 = P_2 = P_3 = P_4 = \frac{1}{4};$$

$$\lambda_1 = -\lambda_4;\ \lambda_2 = -\lambda_3;$$

$$\lambda_1^2 + \lambda_2^2 = \frac{D}{2P}.$$

The system (4.167) does not enable us to determine all the discrete states uniquely. Let $\lambda_2 = -0.5$. Then when D = 1, we get

$$\lambda_1 = -\sqrt{1.75} \approx -1.32.$$

That is, $\lambda_1 = -1.32$, $\lambda_2 = -0.5$, $\lambda_3 = 0.5$, and $\lambda_4 = 1.32$.

In these transformations all the discrete states are equiprobable, since $P_1 = P_2 = P_3 = P_4$. /150

We can set up the problem of determining the discrete states and their probabilities from the condition of satisfying the moments higher than the second order, for example, normal distribution. Here we will have for n = 2:

$$\left.\begin{aligned} M[\Lambda^{2k+1}] &= \lambda_1^{2k+1} P_1 + \lambda_2^{2k+1} P_2 = 0 \quad (k = 0, 1, 2, \ldots); \\ M[\Lambda^2] &= \lambda_1^2 P_1 + \lambda_2^2 P_2 = D. \end{aligned}\right\}$$

Hence follows the solution obtained by:

$$\lambda_1 = -\lambda_2 = \sqrt{D};\quad P_1 = P_2 = \frac{1}{2}.$$

For n = 3, we have the following system of equations:

$$
\begin{gathered}
M\left[\Lambda^{2k+1}\right] = 0;\ k = 0, 1, 2, \ldots; \\
M\left[\Lambda^{2}\right] = \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3 = D = \sigma^2; \\
M\left[\Lambda^{4}\right] = \lambda_1^4 P_1 + \lambda_2^4 P_2 + \lambda_3^4 P_3 = 3\sigma^4; \\
P_1 + P_2 + P_3 = 1,
\end{gathered}
$$

whence it follows that

$$
\begin{gathered}
\lambda_2 = 0,\ \lambda_1 = -\lambda_3;\ P_1 = P_3; \\
2\lambda_1^2 P_1 = \sigma^2; \\
2\lambda_1^4 P_1 = 3\sigma^4; \\
2P_1 + P_3 = 1
\end{gathered}
$$

or

$$
\begin{gathered}
P_1 = P_3 = \frac{1}{6};\quad P_2 = \frac{2}{3}; \\
\lambda_1 = \sqrt{3}\,\sigma;\ \lambda_2 = 0;\ \lambda_3 = \sqrt{3}\,\sigma.
\end{gathered}
$$

When $n > 3$, we can easily perform similar transformations. The results of calculations up to n = 5 are given in Tables 4.22 - 4.24.

TABLE 4.22. DISCRETE STATES

Number of states	1	2	3	4	5
1	—	—	—	—	—
2	σ	$-\sigma$	—	—	—
3	$\sqrt{3}\sigma$	0	$-\sqrt{3}\sigma$	—	—
4	$\sqrt{6-\frac{1}{\sqrt{2}}}\,\sigma$	$\frac{1}{\sqrt{2}}\sigma$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{6-\frac{1}{\sqrt{2}}}\,\sigma$	—
5	$\sqrt{5+\sqrt{10}}\sigma$	$\sqrt{5-\sqrt{10}}\sigma$	0	$-\sqrt{5-\sqrt{10}}\sigma$	$-\sqrt{5+\sqrt{10}}\sigma$

TABLE 4.23. PROBABILITIES OF STATES /151

Number of states	1	2	3	4	5
1	—	—	—	—	—
2	$\frac{1}{2}$	$\frac{1}{2}$	—	—	—
3	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	—	—
4	$\frac{5-2\sqrt{2}}{88}$	$\frac{39+2\sqrt{2}}{88}$	$\frac{39+2\sqrt{2}}{88}$	$\frac{5-2\sqrt{2}}{88}$	—
5	$\frac{7-2\sqrt{10}}{60}$	$\frac{7+2\sqrt{10}}{60}$	$\frac{8}{15}$	$\frac{7+2\sqrt{10}}{60}$	$\frac{7-2\sqrt{10}}{60}$

TABLE 4.24. MOMENTS OF THE k-th ORDER

Moment number	Normal distr.	Number of states			
		2	3	4	5
1	0	0	0	0	0
2	σ^2	σ^2	σ^2	σ^2	σ^2
3	0	0	0	0	0
4	$3\sigma^4$	σ^4	$3\sigma^4$	$3\sigma^4$	$3\sigma^4$
5	0	0	0	0	0
6	$15\sigma^6$	σ^6	$9\sigma^6$	$15\sigma^6$	$15\sigma^6$
7	0	0	0	0	0

Table 4.22 contains the numerical values of discrete states of the random discrete variable as a function of the number of states. Table 4.23 gives the probabilities of the discrete states, and Table 4.24 gives the numerical values of the moments for the discrete random variable and the continuous random variable with normal distribution.

When the number of states n and $\mathbb{D}_i = 1$ $(i = 1, 2, \ldots, m)$, the probability of each state of sequence (4.159) will be equal to n^{-m}, and the number of all states -- $N = n^m$. When there are sufficiently large *m* and *n*, the number of all states N of sequence (4.159) will be sufficiently large, which represents serious difficulties in investigating the scatter of the motion of flight vehicles in the earth's atmosphere. Therefore we have the necessity of examining the possibility of obtaining a discrete-continuous noncanonical representation of atmospheric perturbations in the form /152

$$\xi(t) = \Lambda_1 \cos \Omega^* t + \Lambda_2 \sin \Omega^* t,$$

where Λ_1 and Λ_2 are random variables of the discrete type with the characteristics:

$$M[\Lambda_1] = M[\Lambda_2] = 0;$$
$$M[\Lambda_1^2] = M[\Lambda_2^2] = 1;$$
$$M[\Lambda_1\Lambda_2] = 0;$$

Ω^* is a random variable.

Considering the results obtained in Section 4.2, to determine the characteristics of the random variable Ω^* we will have the identity

$$r_\xi(t) = M[\cos \Omega^* t]. \tag{4.168}$$

If $\Omega^* = \Lambda$, that is, a random variable of the discrete type with the number of levels equal to n, Eq. (4.168) becomes

$$r_\xi(t) = \sum_{i=1}^{n} \cos \lambda_i t P_i. \qquad (4.169)$$

Since $P_i > 0$ and $\sum_{i=1}^{n} P_i = 1$, the problem of determining the quantities of the states λ_i and their probabilities by using the expression is considerably more difficult. Therefore it appears possible to use the following procedure. Let the distribution of the random frequency Ω^* be found from the condition (4.168), then by replacing the continuous random variable with the discrete random variable with a finite number of discrete states and with certain probabilities of these states, as was done above for the normal distribution of a random variable, we can satisfy Eq. (4.169) with a specified error.

The representations of atmospheric perturbations considered in Sections 4.2 - 4.6 essentially are the mathematical models constructed on the basis of the same statistical characteristics of the atmospheric parameters we have considered. In addition to the different rate of convergence, the difference between them is that if the canonical expressions (4.68) and the expressions in the principal components (4.158) can be constructed for any of the set of the profile of the physical atmospheric parameter considered, then the profiles of physical atmospheric parameters modelled by using the spectral expansions (canonical and noncanonical) are selected randomly by means of the realizations of the above-indicated random numbers.

The selection of a particular model for solving problems of controlling the motion of flight vehicles in the dense atmospheric layers must be determined by the nature of the problem formulated.

CHAPTER FIVE /153

STATISTICAL METHODS OF INVESTIGATING THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS

5.1. Models of Nonlinear Control Processes

When designing automatic control systems of flight vehicles moving in the earth's atmosphere, it becomes necessary to take into consideration the effect of fluctuations in atmospheric parameters on the scatter of trajectories. Allowing for fluctuations in the thermodynamic parameters of the atmosphere (density $\Delta\rho$, wind w, temperature Δt, and pressure Δp), which are random functions, in the problems of the control of flight vehicles in the earth's atmosphere can be based on methods of statistical analysis and the synthesis of dynamic stochastic systems developed on the basis of the general theory of random functions [2, 5, 16, 17, 25, 26, 27, 38, 48].

By flight vehicle control we mean the control of the motion of its mass center and motion about the mass center.

Both the motion of the mass center of flight vehicles as well as the motion about the mass center are described by nonlinear differential equations of the form

$$\dot{X} = F(X, U\xi, t), \quad X(t_0) = X_0, \tag{5.1}$$

where X is the n-dimensional vector of the phase coordinates of the flight vehicles in the coordinate system selected; t is the instantaneous time; ξ is the 1-dimensional vector of the perturbing actions, including functions characterizing the fluctuations in the thermodynamic parameters of the atmosphere; X_0 is the n-dimensional vector of the initial conditions of phase coordinates; and U is the r-dimensional vector of forces or moments controlling flight vehicle motion.

Depending on the type of the flight vehicle, its function, and its design and aerodynamic characteristics, one of the models given in the studies /13, 79/ can be taken as the system of equations (5.1). In the introduction, equations (1) - (4) describe the motion of the mass center of a flight vehicle in the earth's atmosphere.

Selection of a coordinate system in the form of mathematical model of the process of the control of flight vehicle motion usually is dictated by the goal of direct investigations. Therefore, primary attention must be concentrated on setting forth the statistical methods of investigating processes described by nonlinear stochastic equations of the form (5.1). /154

Using the material from Chapter Four, we can examine five forms in which the fluctuations of thermodynamic parameters of the atmosphere obtained within the framework of correlation theory can be represented /2, 60, 79/:

1) random functions;

2) canonical representations of random functions with continuous random elements;

3) noncanonical representations of random functions with continuous random elements;

4) canonical and noncanonical representations of random functions with discrete-continuous elements; and

5) representation of random functions by transformation of white noise using shaping filters.

Here the canonical representations of fluctuations in thermodynamic parameters of the atmosphere can be:

expansions of random functions by elements obtained from using the orthogonalization process (after Pugachev);

spectral expansions of random functions; and

expansions of random functions in eigenfunctions (method of component analysis), and so on.

Depending on the kind of random function model used, different models of the control process (5.1) can be constructed. Of interest are the following models of control processes differing by methods of statistical analysis:

1. The general model of a stochastic process described by the vector nonlinear differential equation (5.1) in which fluctuations in atmospheric parameters are random functions with assigned statistical characteristics $M[\xi(t)], M[\xi(t)\xi^*(t)]$, and so on.

2. The model of a stochastic process using representations of random functions in the form of canonical or noncanonical expansions with continuous random variables and described by the vector nonlinear differential equation

$$\dot{X} = F(t, V, U, X), \ X(t_0) = \mathbf{X}_0, \tag{5.2}$$

where V is the m-dimensional vector of noncorrelated continuous random variables with an assigned law of distribution of probability density.

3. A model of a stochastic process using representations of random functions in the form of canonical or noncanonical expansions with uncorrelated discrete random variables and described by the vector nonlinear differential equation

$$\dot{X} = F(X, t, U, \Lambda), \ X(t_0) = X_0, \tag{5.3}$$

where Λ is the m-dimensional vector of discrete random variables /155
with assigned distribution of discrete-state probabilities.

4. A model of a stochastic process using the differential model of the "white noise" filter $\psi(t)$ and described by the nonlinear differential equation

$$\dot{X} = F(X, t, \psi(t), U), \ X(t_0) = X_0, \tag{5.4}$$

where $\psi(t)$ is the s-dimensional "white noise" vector with a zero-valued mathematical expectation $M[\psi(t)] = 0$ and a specified correlation function of the form

$$M[\psi(t)\psi^*(\tau)] = S(t)\delta(t - \tau).$$

Above the following notation was used: S(t) is the matrix of the spectral density of "white noise" and $\delta(t - \tau)$ is the delta-function.

In model (5.4), vector X includes also phase coordinates of the "white noise" filter.

For each of the mathematical models (5.1) - (5.4), we must examine the possible directions of the statistical analysis of control processes for an assigned vector of controls U defining the control process.

One of the possible approaches to analyzing nonlinear stochastic processes is the method of complete linearization of nonlinear equations of the form (5.1) if linearization is possible. The idea of linearization is based on the assumed, and usually attainable smallness of the deviations from the reference process realized by introducing control with respect to the deviations.

Essentially, the method of complete linearization is as follows. We can represent the solutions to differential equation (5.1) in the form

$$X(t)=\tilde{X}(t)+\Delta X(t), \tag{5.5}$$

where $\tilde{X}$ is the vector of the reference solutions of nonlinear equations (5.1) obtained by integrating the latter for zero perturbing actions $\xi(t)$ and assigned initial conditions $X(t_0)=\tilde{X}_0$; $\Delta X(t)$ is the vector of deviations of the solutions of the perturbed system (5.1) from the reference solutions caused by the presence in the right side of the equations (5.1) of perturbations $\xi(t)$.

By carrying out the complete linearization of equations (5.1), we can obtain a linear model of the process of the form

$$\Delta\dot{X}=A(t)\Delta X+C(t)\xi(t),\ \Delta X(t_0)=\Delta X_0, \tag{5.6}$$

where

$$A(t)=\left.\frac{\partial F}{\partial X}\right|_{\tilde{X}}$$

is a matrix of order (n, n) computed for the reference solutions X(t) and characterizing the properties of the process;

$$C(t)=\left.\frac{\partial F}{\partial \xi}\right|_{\tilde{X}}$$

is a matrix of order (n, 1) computed for the reference solutions /156
$\tilde{X}(t)$ and characterizing the degree to which the perturbing actions affect the process under study.

Eq. (5.6) is a linear model of the nonlinear process (5.1) for a specified control U(X, t). In the linear model (5.6), the perturbation vector can be represented in any of the above-listed methods of representing random functions.

Of interest is the linear model for the process (5.1) if the control is organized so as to reduce the numerical values of the vector of deviations $\Delta X(t)$ from the zero value. In this case, the control vector can be represented in the form

$$U(X, t) = U(t, \overline{\Delta X}). \qquad (5.7)$$

Then the linear model of process (5.1) has the following form:

$$\begin{gathered} \Delta\dot{X} = A(t)\Delta X + B(t)U(\Delta X, t) + C(t)\xi(t), \\ \Delta X(t_0) = \Delta X_0, \end{gathered} \qquad (5.8)$$

where $B(t) = \frac{\partial F}{\partial U}\Big|_X$ is a matrix of order (n, r) characterizing the effectiveness of the control actions $\tilde{U}$.

Linear models (5.6) and (5.8) are set up on the assumption that the vector of the reference controls $\tilde{U}$ is equal to zero. If the vector of the reference controls $\tilde{U}$ is not equal to zero, but the control can berrepresented as the two summands

$$U(\Delta X, t) = \widetilde{U}(t) + \Delta U(t, \Delta X), \qquad (5.9)$$

then the reference solution is usually obtained by solving the nonlinear differential equations (5.1) for $U = \tilde{U}(t)$, and the model of process (5.8) is of the form:

$$\begin{gathered} \Delta\dot{X} = A(t)\Delta X + B(t)\Delta U + C(t)\xi(t), \\ \Delta X(t_0) = \Delta X_0. \end{gathered} \qquad (5.10)$$

In practical applications, we are interested in the case of control when the controlling function $\Delta U(t, \Delta X)$ is replaced by the controlling function $\Delta U(\mu, \Delta X)$, that is, the process of forming the control by vector $\Delta X(t)$ takes place with respect to a certain function

$$\mu = \varphi(X, U, \xi, t), \qquad (5.11)$$

which is a function of phase coordinates, controls, perturbations, and time.

This problem is examined in detail in /59/ for the case when the function μ depends on the phase coordinates of the process. The use of measurements in the body-axes coordinate system leads to the necessity of examining the control processes in which the argument of the control action is determined, in addition, by perturbations and control actions.

The requirement of monotonicity is imposed on the function $\varphi(X, U, \xi, t)$. Realizing control with respect to the function μ /157 means that the independent variable (time t) in differential equation (5.1) must be replaced with another independent variable (function μ). In this case the differential equation of process (5.1) can be represented as

$$\frac{dX}{d\mu} = \frac{F(X, U, t, \xi)}{\dot{\varphi}(X, \xi, U, t)}, \quad X(\mu_0) = X_0, \tag{5.12}$$

and the control process must be supplemented with a differential equation for determining the instantaneous time of the process

$$\frac{dt}{d\mu} = \frac{1}{\dot{\varphi}(X, t, U, \xi)}, \quad t(\mu_0) = t_0. \tag{5.13}$$

Carrying out the complete linearization of equations (5.12) and (5.13), we get the linear model of the process in the following form:

$$\frac{d\Delta X}{d\mu} = D(\mu)\Delta X + P(\mu)\Delta U + Q(\mu)\xi + h(\mu)\Delta t, \tag{5.14}$$

$$\Delta X(\mu_0) = \Delta X_0,$$

$$\frac{d\Delta t}{d\mu} = \alpha(\mu)\Delta X + \beta(\mu)\Delta U + \gamma(\mu)\xi + \delta(\mu)\Delta t, \tag{5.15}$$

$$\Delta t(\mu_0) = \Delta t_0,$$

where

$$D(\mu) = \frac{\partial}{\partial X}\left(\frac{F}{\dot{\varphi}}\right); \quad P(\mu) = \frac{\partial}{\partial U}\left(\frac{F}{\dot{\varphi}}\right);$$

$$Q(\mu) = \frac{\partial}{\partial \xi}\left(\frac{F}{\dot{\varphi}}\right); \quad h(\mu) = \frac{\partial}{\partial t}\left(\frac{F}{\dot{\varphi}}\right);$$

$$\alpha(\mu) = \frac{\partial}{\partial X}\left(\frac{1}{\dot{\varphi}}\right); \quad \beta(\mu) = \frac{\partial}{\partial U}\left(\frac{1}{\dot{\varphi}}\right);$$

$$\gamma(\mu) = \frac{\partial}{\partial \xi}\left(\frac{1}{\dot{\varphi}}\right); \quad \delta(\mu) = \frac{\partial}{\partial t}\left(\frac{1}{\dot{\varphi}}\right) \tag{5.16}$$

are the matrices of the linearization coefficients calculated for the reference solutions of equations (5.1), when $X(\mu_0) = \tilde{X}_0$.

It should be noted that $\tilde{U}(t) = \tilde{U}(\mu)$, since the unperturbed motion described by system (5.1) when $\xi(t) = 0$ changes identically both with respect to time as well as in coordinates μ.

The function μ is called the parameter of the control process /59/, equations (5.12) and (5.13) are called nonlinear parametric equations, and equations (5.14) and (5.15) are called linear models of the nonlinear parametric equations.

Introducing the matrices /158

$$\varepsilon = \left(\frac{D}{\alpha}\,\frac{h}{\delta}\right),\quad S = \left(\frac{P}{\beta}\right),\quad L = \left(\frac{Q}{\gamma}\right)$$

and the vector of order (n + 1)

$$Y = (\Delta X, \Delta t),$$

we can represent the system of equations (5.14) and (5.15) in vector form

$$\frac{dY}{d\mu} = \varepsilon(\mu)Y + S(\mu)\Delta U + L(\mu)\xi(\mu),$$
$$Y(\mu_0) = Y_0. \tag{5.17}$$

Matrices (5.16) can be computed based on the above-presented formulas, by first writing out equations (5.12) and (5.13) in particular cases of parameter μ. However, the linearization process is a quite cumbersome procedure, therefore we must consider the algorithm of searching for a relationship between matrices D, $P, Q, h, \alpha, \beta, \gamma$ and δ and the matrices of the linearization coefficients A, B, and C. To do this, let us determine the partial derivatives of (5.16). Obviously,

$$\begin{aligned}
D &= \frac{\partial}{\partial X}\left(\frac{F}{\dot\varphi}\right) = \frac{1}{\dot\varphi^2}\left[\frac{\partial F}{\partial X}\dot\varphi - F\frac{\partial\dot\varphi}{\partial X}\right],\\
P &= \frac{\partial}{\partial U}\left(\frac{F}{\dot\varphi}\right) = \frac{1}{\dot\varphi^2}\left[\frac{\partial F}{\partial U}\dot\varphi - F\frac{\partial\dot\varphi}{\partial U}\right],\\
Q &= \frac{\partial}{\partial \xi}\left(\frac{F}{\dot\varphi}\right) = \frac{1}{\dot\varphi^2}\left[\frac{\partial F}{\partial \xi}\dot\varphi - F\frac{\partial\dot\varphi}{\partial \xi}\right],\\
\alpha &= \frac{\partial}{\partial X}\left(\frac{1}{\dot\varphi}\right) = -\frac{1}{\dot\varphi^2}\frac{\partial\dot\varphi}{\partial X},\\
\beta &= \frac{\partial}{\partial U}\left(\frac{1}{\dot\varphi}\right) = -\frac{1}{\dot\varphi^2}\frac{\partial\dot\varphi}{\partial U},\\
\gamma &= \frac{\partial}{\partial \xi}\left(\frac{1}{\dot\varphi}\right) = -\frac{1}{\dot\varphi^2}\frac{\partial\dot\varphi}{\partial \xi},\\
\delta &= \frac{\partial}{\partial t}\left(\frac{1}{\dot\varphi}\right) = -\frac{1}{\dot\varphi^2}\frac{\partial\dot\varphi}{\partial t},\\
h &= \frac{\partial}{\partial t}\left(\frac{F}{\dot\varphi}\right) = \frac{1}{\dot\varphi^2}\left[\frac{\partial F}{\partial t}\dot\varphi - F\frac{\partial\dot\varphi}{\partial t}\right].
\end{aligned} \tag{5.18}$$

[1]An algorithm was derived in collaboration with Yu. B. Kornilov.

Since $A=\frac{\partial F}{\partial X}, B=\frac{\partial F}{\partial U}$, and $C=\frac{\partial F}{\partial \xi}, \Gamma=\frac{\partial F}{\partial t}$, equations (5.18) /159 will become:

$$
\begin{aligned}
D &= \frac{1}{\dot{\varphi}^2}\left[A\dot{\varphi}-F\frac{\partial\dot{\varphi}}{\partial X}\right],\\
P &= \frac{1}{\dot{\varphi}^2}\left[B\dot{\varphi}-F\frac{\partial\dot{\varphi}}{\partial U}\right],\\
Q &= \frac{1}{\dot{\varphi}^2}\left[C\dot{\varphi}-F\frac{\partial\dot{\varphi}}{\partial \xi}\right],\\
h &= \frac{1}{\dot{\varphi}^2}\left[\Gamma\dot{\varphi}-F\frac{\partial\dot{\varphi}}{\partial t}\right],\\
\alpha &= -\frac{1}{\dot{\varphi}^2}\frac{\partial\dot{\varphi}}{\partial X};\quad \beta=-\frac{1}{\dot{\varphi}^2}\frac{\partial\dot{\varphi}}{\partial U},\\
\gamma &= -\frac{1}{\dot{\varphi}^2}\frac{\partial\dot{\varphi}}{\partial \xi},\quad \delta=-\frac{1}{\dot{\varphi}^2}\frac{\partial\dot{\varphi}}{\partial t}.
\end{aligned}
\tag{5.19}
$$

Expressions (5.19) can be transformed if we use the total derivative of the function $\dot{\varphi}$ in the form

$$
\begin{aligned}
\dot{\varphi} &= \frac{\partial\varphi}{\partial X}\dot{X}+\frac{\partial\varphi}{\partial U}\dot{U}+\frac{\partial\varphi}{\partial \xi}\dot{\xi}+\frac{\partial\varphi}{\partial t}=\frac{\partial\varphi}{\partial X}F+\\
&+\frac{\partial\varphi}{\partial U}\dot{U}+\frac{\partial\varphi}{\partial \xi}\dot{\xi}+\frac{\partial\varphi}{\partial t},
\end{aligned}
\tag{5.20}
$$

where $\dot{U}$ is the derivative of the reference control action $\tilde{U}$; $\dot{\xi}$ is the derivative of the vector of the atmospheric perturbing actions, which were taken into account in computing the reference motion $\tilde{X}(t)$.

Denoting $\dot{\tilde{U}}(t)=g(t), \dot{\tilde{\xi}}(t)=q(t)$, we can represent expresseion (5.20) in the form

$$
\dot{\varphi}=\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial U}g+\frac{\partial\varphi}{\partial \xi}q+\frac{\partial\varphi}{\partial t}.
\tag{5.21}
$$

To determine the matrices D, P, Q, n, α, β, γ, and δ, we must compute in the general form the partial derivatives $\frac{\partial\dot{\varphi}}{\partial X}, \frac{\partial\dot{\varphi}}{\partial U}, \frac{\partial\dot{\varphi}}{\partial \xi}, \frac{\partial\dot{\varphi}}{\partial t}$. This can easily be done if we use the following rules for obtaining partial derivatives of complex functions:

the derivative of scalar H with respect to a vector is the vector-row

$$\frac{\partial a}{\partial X} = \left(\frac{\partial a}{\partial x_1} \frac{\partial a}{\partial x_2} \cdots \frac{\partial a}{\partial x_n}\right),$$

the derivative of a scalar product with respect to a vector is the vector-row /160

$$\frac{d}{dX}(a, b) = \frac{d}{dX}(a^* b) = a^* \frac{db}{dX} + b^* \frac{da}{dX},$$

a and b are columns, (da/dX and db/dX are matrices); and

the second derivative of a scalar with respect to a vector is the matrix

$$\frac{d^2 a}{dX^2} = \begin{bmatrix} \frac{d^2 a}{dx_1^2} & \frac{d^2 a}{dx_1 dx_2} & \cdots & \frac{d^2 a}{dx_1 dx_n} \\ \frac{d^2 a}{dx_1 dx_2} & \frac{d^2 a}{dx_2^2} & \cdots & \frac{d^2 a}{dx_2 dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^2 a}{dx_1 dx_n} & \frac{d^2 a}{dx_2 dx_n} & \cdots & \frac{d^2 a}{dx_n^2} \end{bmatrix}.$$

Following these rules, we will have:

$$\frac{\partial \dot{\varphi}}{\partial X} = F^* \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial \varphi}{\partial X} \frac{\partial F}{\partial X} + g^* \frac{\partial^2 \varphi}{\partial U \partial X} + \frac{\partial \varphi}{\partial U} \frac{\partial g}{\partial X} + q^* \frac{\partial^2 \varphi}{\partial \xi \partial X} + \frac{\partial \varphi}{\partial \xi} \frac{\partial q}{\partial X} + \frac{\partial^2 \varphi}{\partial t \partial X};$$

$$\frac{\partial \dot{\varphi}}{\partial U} = F^* \frac{\partial^2 \varphi}{\partial X \partial U} + \frac{\partial \varphi}{\partial X} \frac{\partial F}{\partial U} + g^* \frac{\partial^2 \varphi}{\partial U^2} + \frac{\partial \varphi}{\partial U} \frac{\partial g}{\partial U} + q^* \frac{\partial^2 \varphi}{\partial \xi \partial U} + \frac{\partial \varphi}{\partial \xi} \frac{\partial q}{\partial U} + \frac{\partial^2 \varphi}{\partial t \partial U};$$

$$\frac{\partial \dot{\varphi}}{\partial \xi} = F^* \frac{\partial^2 \varphi}{\partial X \partial \xi} + \frac{\partial \varphi}{\partial X} \frac{\partial F}{\partial \xi} + g^* \frac{\partial^2 \varphi}{\partial U \partial \xi} + \frac{\partial \varphi}{\partial U} \frac{\partial g}{\partial \xi} + q^* \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial \varphi}{\partial \xi} \frac{\partial q}{\partial \xi} + \frac{\partial^2 \varphi}{\partial t \partial \xi};$$

$$\frac{\partial \dot{\varphi}}{\partial t} = \frac{\partial^2 F}{\partial X \partial t} F + \frac{\partial \varphi}{\partial X} \frac{\partial F}{\partial t} + \frac{\partial^2 \varphi}{\partial U \partial t} g + \frac{\partial \varphi}{\partial U} \frac{\partial g}{\partial t} + \frac{\partial^2 \varphi}{\partial \xi \partial t} q + \frac{\partial \varphi}{\partial \xi} \frac{\partial q}{\partial t} + \frac{\partial^2 \varphi}{\partial t^2}.$$

These expressions were obtained by determining the partial derivatives with respect to vectors X, U, ξ, and time t from the first part of expression (5.20). They become considerably simplified if we consider that

$$\frac{\partial g}{\partial X} = \frac{\partial q}{\partial X} = \frac{\partial g}{\partial U} = \frac{\partial q}{\partial U} = \frac{\partial g}{\partial \xi} = \frac{\partial q}{\partial \xi} = 0,$$

and introduce the notation

$$\frac{\partial g}{\partial t} = \dot{g} = g_1; \quad \frac{\partial q}{\partial t} = \dot{q} = q_1, \quad \Gamma = \frac{\partial F}{\partial t}.$$

We get

$$\frac{\partial\dot{\varphi}}{\partial X}=F^*\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial\varphi}{\partial X}A+g^*\frac{\partial^2\varphi}{\partial U\partial X}+q^*\frac{\partial^2\varphi}{\partial X\partial\xi}+\frac{\partial^2\varphi}{\partial t\partial X};$$
$$\frac{\partial\dot{\varphi}}{\partial U}=F^*\frac{\partial^2\varphi}{\partial X\partial U}+\frac{\partial\varphi}{\partial X}B+g^*\frac{\partial^2\varphi}{\partial U^2}+q^*\frac{\partial^2\varphi}{\partial U\partial\xi}+\frac{\partial^2\varphi}{\partial t\partial U};$$
$$\frac{\partial\dot{\varphi}}{\partial\xi}=F^*\frac{\partial^2\varphi}{\partial X\partial\xi}+\frac{\partial\varphi}{\partial X}C+g^*\frac{\partial^2\varphi}{\partial U\partial\xi}+q^*\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial t\partial\xi};$$
$$\frac{\partial\dot{\varphi}}{\partial t}=\frac{\partial^2\varphi}{\partial X\partial t}F+\frac{\partial\varphi}{\partial X}\Gamma+\frac{\partial^2\varphi}{\partial U\partial t}g+\frac{\partial\varphi}{\partial U}g_1+\frac{\partial^2\varphi}{\partial\xi\partial t}q+\frac{\partial\varphi}{\partial\xi}q_1+\frac{\partial^2\varphi}{\partial t^2}. \quad (5.22)$$

Substituting expressions (5.21) and (5.22) into equations (5.19), we get working formulas for computing the linearization coefficients of the nonlinear parametric equations (5.12) and (5.13), by employing matrices A(t), B(t), and C(t) of the linear model (5.10), and the partial derivatives of the function ϕ with respect to vectors X, U, ξ, and time t computed for the reference solution $\tilde{X}(t)$..

This algorithm enables us, when model (5.10) is available, to construct quite easily the linear model (5.17) for any function ϕ and given the presence of the reference solution $\tilde{X}(t)$.

Let us examine several particular cases. If we consider $\mu=\varphi(X, U, t)$, formulas (5.22) become

$$\frac{\partial\dot{\varphi}}{\partial X}=F^*\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial\varphi}{\partial X}A+g^*\frac{\partial^2\varphi}{\partial U\partial X}+\frac{\partial^2\varphi}{\partial t\partial X};$$
$$\frac{\partial\dot{\varphi}}{\partial U}=F^*\frac{\partial^2\varphi}{\partial X\partial U}+\frac{\partial\varphi}{\partial X}B+g^*\frac{\partial^2\varphi}{\partial U^2}+\frac{\partial^2\varphi}{\partial t\partial U};$$
$$\frac{\partial\dot{\varphi}}{\partial\xi}=\frac{\partial\varphi}{\partial X}C;$$
$$\frac{\partial\dot{\varphi}}{\partial t}=\frac{\partial^2\varphi}{\partial X\partial t}F+\frac{\partial\varphi}{\partial X}\Gamma+\frac{\partial^2\varphi}{\partial U\partial t}g+\frac{\partial^2\varphi}{\partial U\partial t}g_1+\frac{\partial^2\varphi}{\partial t^2}. \quad (5.23)$$

For $\mu=\varphi(X, t)$, formulas (5.22) are simplified as follows:

$$\frac{\partial\dot{\varphi}}{\partial X}=F^*\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial\varphi}{\partial X}A+\frac{\partial^2\varphi}{\partial t\partial X};$$
$$\frac{\partial\dot{\varphi}}{\partial U}=\frac{\partial\varphi}{\partial X}B;\quad \frac{\partial\dot{\varphi}}{\partial\xi}=\frac{\partial\varphi}{\partial X}C;$$
$$\frac{\partial\dot{\varphi}}{\partial t}=\frac{\partial^2\varphi}{\partial X\partial t}F+\frac{\partial\varphi}{\partial X}\Gamma+\frac{\partial^2\varphi}{\partial t^2}. \quad (5.24)$$

Finally, when $\mu=\varphi(X)$, we get /162

$$\frac{\partial\dot{\varphi}}{\partial X}=F^{*}\frac{\partial^{2}\varphi}{\partial X^{2}}+\frac{\partial\varphi}{\partial X}A;$$
$$\frac{\partial\dot{\varphi}}{\partial U}=\frac{\partial\varphi}{\partial X}B;$$
$$\frac{\partial\dot{\varphi}}{\partial t}=\frac{\partial\varphi}{\partial X}\Gamma;\quad \frac{\partial\dot{\varphi}}{\partial\xi}=\frac{\partial\dot{\varphi}}{\partial X}C. \tag{5.25}$$

For example, substituting expressions (5.24) into equations (5.19) with $\dot{\varphi}=\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}$ for the parameter $\mu=\varphi(X, t)$, we get the following working formulas:

$$D=\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[A\frac{\partial\varphi}{\partial X}F+A\frac{\partial\varphi}{\partial t}-FF^{*}\frac{\partial^{2}\varphi}{\partial X^{2}}-F\frac{\partial\varphi}{\partial X}A-F\frac{\partial^{2}\varphi}{\partial t\partial X}\right];$$
$$P=\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[B\frac{\partial\varphi}{\partial X}F+B\frac{\partial\varphi}{\partial t}-F\frac{\partial\varphi}{\partial X}B\right];$$
$$Q=\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[C\frac{\partial\varphi}{\partial X}F+C\frac{\partial\varphi}{\partial t}-F\frac{\partial\varphi}{\partial X}C\right];$$
$$h=\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[\Gamma\frac{\partial\varphi}{\partial X}F+\Gamma\frac{\partial\varphi}{\partial t}-F\frac{\partial^{2}\varphi}{\partial X\partial t}F-F\frac{\partial\varphi}{\partial X}\Gamma-F\frac{\partial^{2}\varphi}{\partial t^{2}}\right];$$
$$\alpha=-\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[F^{*}\frac{\partial^{2}\varphi}{\partial X^{2}}+\frac{\partial\varphi}{\partial X}A+\frac{\partial^{2}\varphi}{\partial t\partial X}\right];$$
$$\beta=-\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\frac{\partial\varphi}{\partial X}B;$$
$$\gamma=-\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\frac{\partial\varphi}{\partial X}C;$$
$$\delta=-\frac{1}{\left(\frac{\partial\varphi}{\partial X}F+\frac{\partial\varphi}{\partial t}\right)^{2}}\left[\frac{\partial^{2}\varphi}{\partial X\partial t}F+\frac{\partial\varphi}{\partial X}\Gamma+\frac{\partial^{2}\varphi}{\partial t^{2}}\right]. \tag{5.26}$$

Relations (5.26) graphically show the simplicity of transformations from the linear model (5.10) to the linear model (5.17) when implementing controls with respect to some parameter μ, which is a function of phase coordinates and time. Similar relations can be written out also for the other forms of function μ. Naturally, the simplest relations (5.22) are obtained for a parameter of the form $\mu=\varphi(X)$, since here /163

$$
\left.
\begin{aligned}
D &= \frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\left[A\frac{\partial\varphi}{\partial X}F - F\frac{\partial\varphi}{\partial X}A - FF^*\frac{\partial^2\varphi}{\partial X^2}\right];\\
P &= \frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\left[B\frac{\partial\varphi}{\partial X}F - F\frac{\partial\varphi}{\partial X}B\right];\\
Q &= \frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\left[C\frac{\partial\varphi}{\partial X}F - F\frac{\partial\varphi}{\partial X}C\right];\\
h &= \frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\left[\Gamma\frac{\partial\varphi}{\partial X}F - F\frac{\partial\varphi}{\partial X}\Gamma\right];\\
\alpha &= -\frac{2}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\left[F^*\frac{\partial^2\varphi}{\partial X^2} + \frac{\partial\varphi}{\partial X}A\right];\\
\beta &= -\frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\frac{\partial\varphi}{\partial X}B;\\
\gamma &= -\frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\frac{\partial\varphi}{\partial X}C;\\
\delta &= -\frac{1}{\left(\frac{\partial\varphi}{\partial X}F\right)^2}\frac{\partial\varphi}{\partial X}\Gamma.
\end{aligned}
\right\} \quad (5.27)
$$

To prepare for the statistical analysis of models (5.10) and (5.17), we must determine the structure of the control actions.

To do this, let us introduce into consideration a certain system of observational functions

$$
\eta_1(X,t,U,\xi),\quad \eta_2(X,t,U,\xi),\quad \ldots,\quad \eta_k(X,t,U,\xi), \qquad (5.28)
$$

each of which is computed for the reference solution $\tilde{X}(t)$ and is /164
equal to

$$\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_k. \tag{5.29}$$

Using the systems of functions (5.28) and (5.29), let us compute the differences

$$\Delta\eta_1 = \eta_1 - \tilde{\eta}_1, \; \Delta\eta_2 = \eta_2 - \tilde{\eta}_2, \; \ldots, \; \Delta\eta_k = \eta_k - \tilde{\eta}_k \tag{5.30}$$

and let us assume that the control ΔU is a function

$$\Delta U(\Delta\eta_1, \Delta\eta_2, \ldots, \Delta\eta_k). \tag{5.31}$$

Linearizing expressions (5.31), we get

$$\Delta U = G\Delta\eta, \tag{5.32}$$

where $G = \frac{\partial \Delta U}{\partial \Delta\eta}$ is the matrix of order (r, k) characterizing the linear part of the control action.

For the vector $\Delta\eta$, we can also obtain the linear representation in the form

$$\Delta\eta = HY + N\Delta U + M\xi, \tag{5.33}$$

where $H = \frac{\partial \Delta\eta}{\partial Y}$; $N = \frac{\partial \Delta\eta}{\partial U}$; $M = \frac{\partial \Delta\eta}{\partial \xi}$ are linearization matrices of orders (k, (n + 1)), (k, r), and (k, 1), respectively.

From expressions (5.32) and (5.33) we can obtain an expression for the control action in the form

$$\Delta U = (E + GN)^{-1}[GHY + GM\xi]. \tag{5.34}$$

Substituting Eq. (5.34) into Eq. (5.17), we will have

$$\frac{dY}{d\mu} = \tilde{A}(\mu)\,Y + \tilde{C}(\mu)\,\xi(\mu), \tag{5.35}$$

where

$$\tilde{A}(\mu) = \varepsilon + S(E - GN)^{-1}GH,$$
$$\tilde{C}(\mu) = L + S(E - GN)^{-1}GM. \tag{5.36}$$

Thus, the above-considered four nonlinear mathematical models of the control process (5.1) - (5.4) can be supplemented with the linear model (5.35), whose elements in the general case are described by relations (5.36).

Example 5.1. Let us illustrate the foregoing with an illustrative example for a simplified model of the motion of the center of mass of a flight vehicle in the earth's atmosphere, whose equations are presented in /13, 79/. Without considering the earth's rotation, the simplified model of motion of the flight vehicle mass center in the longitudinal plane Oxy (Fig. 5.1) is of the form: /165

$$\left.\begin{aligned} \dot V_x &= -\bar Q_1 \cos(\theta+\alpha) - \bar Y_1 \sin(\theta+\alpha) - \frac{x}{r} g;\\ \dot x &= V_x;\\ \dot V_y &= -\bar Q_1 \sin(\theta+\alpha) + \bar Y_1 \cos(\theta+\alpha) - \frac{R+y}{r} g;\\ \dot y &= V_y; \end{aligned}\right| \quad (5.1.1)$$

$$\left.\begin{aligned} V_x(t_0) = V_{x,0};\ \ V_y(t_0) = V_{y,0};\\ x(t_0) = x_0;\ \ y(t_0) = y_0, \end{aligned}\right| \quad (5.1.2)$$

where the following notation is used:

$$\left.\begin{aligned} r = R + h;\ \ M = \frac{V}{a};\\ \theta = \operatorname{arctg}\frac{V_y}{V_x};\quad g = 9{,}81 = \mathrm{const};\\ h = y + \frac{x^2}{2R};\quad V = \sqrt{V_x^2 + V_y^2};\\ V_x = V\cos\theta;\quad V_y = V\sin\theta;\\ \bar Q_1 = \gamma_0(\alpha, M)\rho(h)V_W^2;\quad \bar Y_1 = \gamma_1(\alpha, M)\rho(h)V_W^2;\\ \gamma_0(\alpha, M) = \frac{C_{x_1}(\alpha, M)S}{2m};\\ \gamma_1(\alpha, M) = \frac{C_{y_1}(\alpha, M)S}{2m}; \end{aligned}\right] \quad (5.1.3)$$

a is the speed of sound; $\rho(h)$ is the density of air; h is the flight altitude of the flight vehicle over the earth's surface; V_W is the airspeed of the flight vehicle; and R is the radius of the earth. /166

The functions $\gamma_0(\alpha, M)$ and $\gamma_1(\alpha, M)$ are determined by design and aerodynamic characteristics of the flight vehicle.

The angle of attack of the flight vehicle is taken as the control action. Let us consider the following by way of perturbations:

deviation of atmospheric density $\rho(h)$ from the standard atmospheric parameters

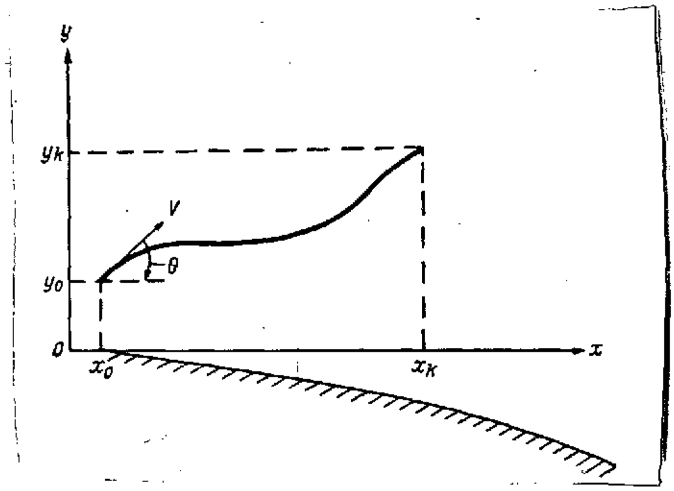


Fig. 5.1. Coordinate System

$$\Delta\rho\ (h) = \rho\,(h) - \rho_0\,(h);$$

deviation of the function γ_0 from the calculated $\tilde{\gamma}_0$

$$\Delta\gamma_0 = \gamma_0 - \tilde{\gamma}_0;$$

deviation of the function γ_1 from the calculated $\tilde{\gamma}_1$

$$\Delta\gamma_1 = \gamma_1 - \tilde{\gamma}_1.$$

The effect of wind perturbations is not taken into account below for methodological considerations, therefore $V_W = V$.

Let the reference control $\tilde{U}(t) = \tilde{\alpha}(t)$ be specified from considerations of the movement of flight vehicle from a point with coordinates $(x_0,\ y_0)$ to a point with coordinates $(x_h,\ y_h)$ and let the reference solution of equations (5.1.1) be obtained given the initial conditions (5.1.2). Linearizing the equations of motion (5.1.1) relative to the reference motions $\tilde{V}_x(t)$, $x(t)$, $\tilde{V}_y(t)$, $y(t)$, we get the linear model of the process in the form

$$\begin{gathered}
\Delta\dot{V}_x = a_{11}\Delta V_x + a_{12}\Delta x + a_{13}\Delta V_y + a_{14}\Delta y + b_1\Delta\alpha + C_{11}\Delta\rho + c_{12}\Delta\gamma_0 + c_{13}\Delta\gamma_1; \\
\Delta\dot{x} = \Delta V_x; \\
\Delta\dot{V}_y = a_{31}\Delta V_x + a_{32}\Delta x + a_{33}\Delta V_y + a_{34}\Delta y + b_3\Delta\alpha + c_{31}\Delta\rho + c_{32}\Delta\gamma_0 + c_{33}\Delta\gamma_1; \\
\Delta\dot{y} = \Delta V_y.
\end{gathered} \tag{5.1.4}$$

Introducing the notation

$$\Delta x = (\Delta V_x, \Delta x, \Delta V_y, \Delta y); \quad \Delta U = \Delta \alpha;$$
$$\xi^* = (\Delta \rho, \Delta \gamma_0, \Delta \gamma_1);$$
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{pmatrix},$$
$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} \\ 0 & 0 & 0 \end{pmatrix},$$

we can represent Eq. (5.1.4) in vector form

$$\Delta \dot{X} = A\Delta X + B\Delta U + C\Delta \xi. \qquad (5.1.5)$$

Let us define the control action ΔU in the form

$$\Delta U = k_1 \Delta V_x + k_2 \Delta x + k_3 \Delta V_y + k_4 \Delta y,$$

or in the vector form /167

$$\Delta U = K^* \Delta X, \qquad (5.1.6)$$

where

$$K^* = (k_1\, k_2\, k_3\, k_4).$$

Substituting Eq. (5.1.6) into Eq. (5.1.5), we get

$$\Delta \dot{X} = (A + BK^*)\, \Delta X + C\xi. \qquad (5.1.7)$$

The process of setting up a linear model for the control of the motion of the flight vehicle mass center can conclude at this point.

Example 5.2. Using the results obtained in example 5.1, let us construct a linear parametric model if the parameter is assigned in the form

$$\mu = \nu_0 x \;(\nu_0 = \mathrm{const}). \qquad (5.2.1)$$

Using Eqs. (5.27), let us find the relations for calculating the parameters of matrices D, P, Q, h, β, γ, and δ of model (5.14) and (5.15). Since

$$\frac{\partial \varphi}{\partial X} = (0\;\nu_0\;0\;0);$$
$$\frac{\partial \varphi}{\partial U} = 0; \quad \frac{\partial \varphi}{\partial \xi} = 0; \quad \frac{\partial \varphi}{\partial t} = 0; \quad \frac{\partial F}{\partial t} = 0;$$
$$F^* = \left(\dot{\tilde{V}}_x \dot{\tilde{x}} \dot{\tilde{V}}_y \dot{\tilde{y}}\right),$$

then the linear model (5.14) and (5.15) can be constructed based on the formulas:

$$
\begin{gathered}
D=\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\left[A\frac{\partial\varphi}{\partial X}F-F\frac{\partial\varphi}{\partial X}A\right];\\
P=\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\left[B\frac{\partial\varphi}{\partial X}F-F\frac{\partial\varphi}{\partial X}B\right];\\
Q=\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\left[C\frac{\partial\varphi}{\partial X}F-F\frac{\partial\varphi}{\partial X}C\right];\\
\alpha=\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\frac{\partial\varphi}{\partial X}A;\quad \beta=-\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\frac{\partial\varphi}{\partial X}B;\quad \gamma=-\frac{1}{\left(\nu_0\dot{\tilde{x}}\right)^2}\frac{\partial\varphi}{\partial X}C;\quad h=\delta=0.
\end{gathered}
\qquad (5.2.2)
$$

Considering that

$$
\frac{\partial\varphi}{\partial X}F=\nu_0\dot{\tilde{x}},
$$

$$
F\frac{\partial\varphi}{\partial X}=\begin{bmatrix}0 & \nu_0 & \dot{\tilde{V}}_x & 0 & 0\\ 0 & \nu_0 & \dot{\tilde{x}} & 0 & 0\\ 0 & \nu_0 & \dot{\tilde{V}}_y & 0 & 0\\ 0 & \nu_0 & \dot{\tilde{y}} & 0 & 0\end{bmatrix},
$$

using expressions (5.2.2) we get /168

$$
\begin{gathered}
D=\begin{bmatrix}
\dfrac{a_{11}\dot{\tilde{x}}-\dot{\tilde{V}}_x}{\nu_0\dot{\tilde{x}}^2} & \dfrac{a_{12}}{\nu_0\dot{\tilde{x}}} & \dfrac{a_{13}}{\nu_0\dot{\tilde{x}}} & \dfrac{a_{14}}{\nu_0\dot{\tilde{x}}}\\
0 & 0 & 0 & 0\\
\dfrac{a_{31}\dot{\tilde{x}}-\dot{\tilde{V}}_y}{\nu_0\left(\dot{\tilde{x}}\right)^2} & \dfrac{a_{32}}{\nu_0\dot{\tilde{x}}} & \dfrac{a_{33}}{\nu_0\dot{\tilde{x}}} & \dfrac{a_{34}}{\nu_0\dot{\tilde{x}}}\\
-\dfrac{\dot{\tilde{y}}}{\nu_0\left(\dot{\tilde{x}}\right)^2} & 0 & \dfrac{1}{\nu_0\dot{\tilde{x}}} & 0
\end{bmatrix};\\
P^*=\left(\frac{b_1}{\nu_0\dot{\tilde{x}}}\quad 0\quad \frac{b_3}{\nu_0\dot{\tilde{x}}}\quad 0\right);\\
Q=\begin{bmatrix}
\dfrac{c_{11}}{\nu_0\dot{\tilde{x}}} & \dfrac{c_{12}}{\nu_0\dot{\tilde{x}}} & \dfrac{c_{13}}{\nu_0\dot{\tilde{x}}}\\
0 & 0 & 0\\
\dfrac{c_{31}}{\nu_0\dot{\tilde{x}}} & \dfrac{c_{32}}{\nu_0\dot{\tilde{x}}} & \dfrac{c_{33}}{\nu_0\dot{\tilde{x}}}\\
0 & 0 & 0
\end{bmatrix};
\end{gathered}
\qquad (5.2.3)
$$

/Continued on following page/

$$\alpha = \left(\frac{1}{v_0 \dot{\tilde{x}}} \quad 0 \quad 0 \quad 0\right);$$
$$\beta = \gamma = 0. \qquad (5.2.3) \text{ (Cont.)}$$

Eqs. (5.2.3) show the considerable simplicity in setting up a linear model of the process (5.14) and (5.15) by using relations (5.27). Summing up the foregoing, let us write out the matrices ε, S, and L, for the linear model of a parametric system:

$$\varepsilon = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & d_{34} & 0 \\ d_{41} & 0 & d_{43} & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \end{bmatrix};$$
$$S = (p_1 \; 0 \; p_3 \; 0 \; 0);$$
$$L = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ 0 & 0 & 0 \\ q_{31} & q_{32} & q_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (5.2.4)$$

Introducing the notation /169

$$Y^* = (\Delta V_x \, \Delta x \, \Delta V_y \, \Delta y \, \Delta t),$$

we can write out the equation of the linear model (5.14) and (5.15) in the form

$$Y' = \varepsilon Y + S\Delta U + L\xi. \qquad (5.2.5)$$

Substituting the control ΔU

$$\Delta U = k_1 \Delta V_x + k_2 \Delta x + k_3 \Delta V_y + k_4 \Delta y + k_5 \Delta t \qquad (5.2.6)$$

or $\Delta U = K^* Y$, where $K^* = (k_1 \; k_2 \; k_3 \; k_4 \; k_5)$, in Eq. (5.2.5) we get

$$Y' = (\varepsilon + SK^*) \, Y + L\xi. \qquad (5.2.7)$$

Thus, the procedure of constructing a linear model for the control process assuming parameter (5.2.1) is completed. We note that in Eq. (5.2.6) the coefficient $k_2 = 0$, since the differential equation for the coordinate Δx in Eq. (5.2.7) is absent (all linearization coefficients in the second row are identically equal to zero).

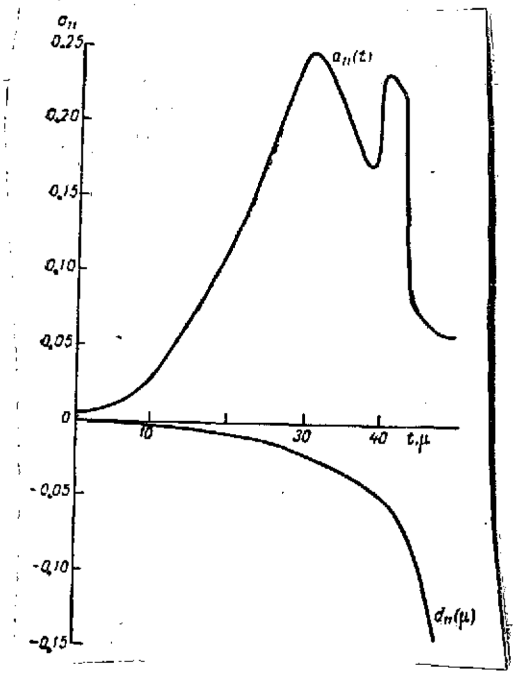


Fig. 5.2. Linearization coefficients $a_{11}(t)$ and $d_{11}(\mu)$

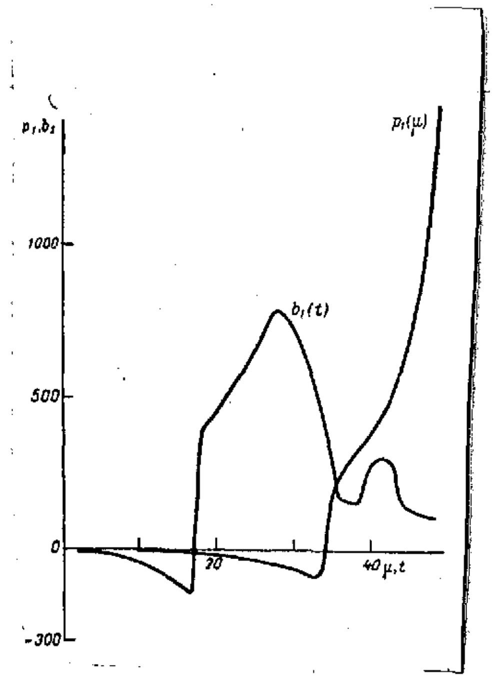


Fig. 5.3. Linearization coefficients $b_1(t)$ and $p_1(\mu)$

Figs. 5.2-5.4 show the variation in several coefficients a_{ij} and d_{ij}, b_i, and p_i as functions of arguments t and μ. These results characterize the variation in the dynamic characteristics of the control object in passing from one control argument t to the other argument $\mu/_0x$.

Constructing the linear model for nonlinear parametric systems with the more complex dependence of parameter μ on phase states V_x, x, V_y, and y also does not occasion serious difficulties.

At this point, the examination of models of control processes can be concluded and we can proceed to considering methods of the statistical analysis of processes for controlling the motion of a flight vehicle in dense atmospheric layers. By the problem of the statistical analysis of control processes we will mean the problem of calculating the mathematical expectations of solutions of Eqs. (5.1) - (5.4), (5.35), and correlation matrices of the solutions /170

$$R_{XX}(t) = M[X(t)X^*(t)]$$

or the statistical characteristics (mathematical expectations and correlation matrices) of several functions of the solutions to the nonlinear stochastic equations.

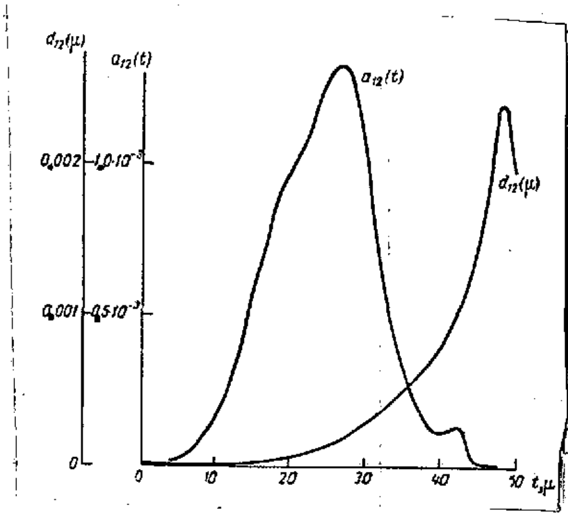


Fig. 5.4. Linearization Coefficients $a_{12}(t)$ and $d_{12}(\mu)$

5.2. Statistical Analysis of Nonlinear Systems Based on a Linear Approximation

Let us write out the solution to Eq. (5.35) for the linear model of a control process in the Cauchy form

$$Y(\mu) = \Phi(\mu) Y_0 + \Phi(\mu) \int_{\mu_0}^{\mu} \Phi^{-1}(\tau) \tilde{C}(\tau) \xi(\tau) d\tau. \quad (5.37)$$

We assume that the matrix of the initial-condition correlations

$$R_{YY}(\mu) = R_0 = M[Y_0 Y_0^*] \quad (5.38)$$

and the correlation matrix of the perturbing actions -- fluctuations of the atmospheric parameters

$$R_{\xi\xi}(t, \tau) = M[\xi(t) \xi^*(\tau)]. \quad (5.39)$$

are given.

Let us find the correlation matrix of the solution /171

$$R_{YY}(\mu, \mu) = M[Y(\mu) Y^*(\mu)],$$

by using the solution of equation (5.35) in the form (5.37). Assuming that the statistical interrelationship between the initial-condition vector Y_0 and the perturbation vector $\xi(\mu)$ is absent, we will have

$$R_{YY}(\mu, \mu) = \Phi(\mu) R_0 \Phi^*(\mu) + \int_{\mu_0}^{\mu}\int_{\mu_0}^{\mu} \Phi(\mu) \Phi^{-1}(t) \times$$
$$\times \tilde{C}(t) R_{\xi\xi}(t, \tau) \tilde{C}^*(\tau) [\Phi^{-1}(\tau)]^* \Phi^*(\mu) dt\, d\tau. \quad (5.40)$$

In Eqs. (5.37) and (5.40), the matrix $\Phi(t)$ is a fundamental matrix of the solutions of the differential equation (5.35)

$$\Phi(t_0) = E. \tag{5.41}$$

Computing the correlation matrix $R_{YY}(\mu, \mu)$ by employing expression (5.40) involves several difficulties, namely: preliminary integration of equation (5.41), necessary computations under the sign of the integrals in Eq. (5.40), and computation of the double integral of the function of two variables. The computational difficulties are considerably diminished if we use different representations of the random function.

Let us examine several particular cases. For the canonical representations of atmospheric perturbations taken in the form

$$\xi(t) = \Theta(t) V, \tag{5.42}$$

where $\Theta(t)$ is the coordinate-function matrix and V is the m-dimensional vector of random variables with normal distribution of probability density and with the assigned characteristics

$$M[V] = 0; \quad M[VV^*] = R_{VV},$$

Eq. (5.40) can be transformed to become

$$R_{YY}(\mu) = \Phi(\mu) R_0 \Phi^*(\mu) + \int_{\mu_0}^{\mu}\int_{\mu_0}^{\mu} \Phi(\mu) \Phi^{-1}(t) \tilde{C}(t) \Theta(t) R_{VV} \times \times \Theta^*(\tau) \tilde{C}^*(\tau) [\Phi^{-1}(\tau)]^* \Phi^*(\mu)\, dt\, d\tau. \tag{5.43}$$

In appearance, Eq. (5.43) is not simpler compared with the earlier-derived solution (5.40). However, after uncomplicated transformations, which yield the expression /172

$$R_{YY}(\mu) = \Phi(\mu) R_0 \Phi^*(\mu) + \int_{\mu_0}^{\mu} \Phi(\mu) \Phi^{-1}(t) \tilde{C}(t) \Theta(t) dt \times \times R_{VV} \int_{\mu_0}^{\mu} \Theta^*(\tau) \tilde{C}^*(\tau) [\Phi^{-1}(\tau)]^* \Phi^*(\mu)\, d\tau, \tag{5.44}$$

its simplicity becomes evident.

Introducing the notation

$$H(\mu) = \int_{\mu_0}^{\mu} \Phi(\mu) \Phi^{-1}(t) \tilde{C}(t) \Theta(t)\, dt,$$

we get for (5.44) the following representation:

$$R_{YY}(\mu) = \Phi(\mu) R_0 \Phi^*(\mu) + H(\mu) R_{VV} H^*(\mu). \tag{5.45}$$

Eq. (5.45) can be used in computing the matrix $R_{YY}(\mu, \mu)$ both with the continuous vector V as well as with its discrete representation.

If the vector $\xi(t)$ is "white noise," then by substituting into Eq. (5.40) the correlation function of perturbation in the form

$$R_{\xi\xi}(t, \tau) = S(t)\delta(t-\tau), \tag{5.46}$$

we can get

$$\begin{aligned} R_{YY}(\mu) &= \Phi(\mu) R_0 \Phi^*(\mu) + \int_{\mu_0}^{\mu}\int_{\mu_0}^{\mu} \Phi(\mu)\Phi^{-1}(t)\tilde{C}(t) \times \\ &\times S(t)\delta(t-\tau)\tilde{C}^*(\tau)\left[\Phi^{-1}(\tau)\right]^*\Phi^*(\mu)\,dt\,d\tau = \\ &= \Phi(\mu) R_0 \Phi^*(\mu) + \int_{\mu_0}^{\mu} \Phi(\mu)\Phi^{-1}(t)\tilde{C}(t) S(t)\tilde{C}^*(t)\,dt. \end{aligned} \tag{5.47}$$

Noncanonical representations of random functions do not simplify the calculation scheme (5.40) for computing the required characteristics.

Using relations (5.40), (5.45), and (5.47) involves quite a cumbersome computational process, since we need to integrate twice: first, to compute the fundamental matrix of solutions $\Phi(\mu)$, and second to determine the component of vector Y caused by the action of atmospheric perturbations. Therefore, in actual statistical analysis of linear systems use is made of correlation equations relating the matrix $R_{YY}(\mu, \mu)$ with the matrices R_0 and $R_{\xi\xi}(t, \tau)$.

Denoting /173

$$\psi = \tilde{C}(\mu)\xi(\mu),$$

let us write Eq. (5.35) in the form

$$Y' = \tilde{A}(\mu) Y + \psi(\mu). \tag{5.48}$$

We can easily show that for any random vector-function $\psi(\mu)$ that exhibits all the derivatives $\psi^{(k)}$, the following correlation system of differential equations corresponding to Eq. (5.48) is valid:

$$
\left.\begin{aligned}
R'_{YY} &= \tilde{A}R_{YY} + R_{YY}\tilde{A}^* + R_{Y\psi} + R^*_{Y\psi},\\
R'_{Y\psi} &= \tilde{A}R_{Y\psi} + R_{\psi\psi} + R_{Y\psi'},\\
R'_{Y\psi'} &= \tilde{A}R_{Y\psi'} + R_{\psi\psi'} + R_{Y\psi''},\\
R'_{Y\psi''} &= \tilde{A}R_{Y\psi''} + R_{\psi\psi''} + R_{Y\psi'''},\\
&\dots\dots\dots\dots\dots\dots\\
R_{Y\psi(k)} &= \tilde{A}R_{Y\psi(k)} + R_{\psi\psi(k)} + R_{Y\psi(k+1)},\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right|
$$

with the initial conditions

$$
\left.\begin{aligned}
R_{YY}(\mu_0) &= R_0;\\
R_{Y\psi}(\mu_0) = R_{Y\psi'}(\mu_0) &= \dots = R_{Y\psi(k)}(\mu_0) = 0.
\end{aligned}\right| \qquad \begin{aligned}(5.49)\\(5.50)\end{aligned}
$$

The system of equations (5.49) associates correlation matrices of the output characteristics of process Y with the correlation matrices of perturbations $R_{\psi\psi}, R_{\psi\psi'}, \dots, R_{\psi\psi^{(k)}}$ and the matrix of the system $\tilde{A}$ under investigation. In the general case, it is virtually impossible to use system (5.49) in view of its infinitude. If it turns out that the solution $R_{Y\psi(k+1)}$ is identically equal to zero, then the system of equations (5.49) will be a finite system of correlational differential equations. This system of equations can be used in the statistical analysis of a process based on the linear approximation, that is, based on linear model (5.35).

Note that in deriving system (5.49) use was made of the following transformations:

$$
\left.\begin{aligned}
R'_{YY} &= M\left[\frac{d}{d\mu}(YY^*)\right] = M[Y'Y^* + YY^{*\prime}] =\\
&= M[(\tilde{A}Y + \psi)Y^* + Y(\tilde{A}Y + \psi)^*] =\\
&= \tilde{A}R_{YY} + R_{YY}\tilde{A}^* + R^*_{Y\psi} + R_{Y\psi},
\end{aligned}\right|
$$

$$
\left.\begin{aligned}
R'_{Y\psi} &= M\left[\frac{d}{d\mu}Y\psi^*\right] = M[Y'\psi^* + Y\psi^{*\prime}] =\\
&= M[(\tilde{A}Y + \psi)\psi^* + Y\psi^{*\prime}] = \tilde{A}R_{Y\psi} + R_{\psi\psi} + R_{Y\psi'}
\end{aligned}\right|
$$

and so on. /174

Let us examine the possibility of obtaining an estimate of the term $R_{Y\psi(k+1)}$. To do this, we obtain a system of differential equations (5.49) by using the solution to Eq. (5.48) in the form

(5.40), after first representing it in the form

$$R_{Y\psi}(\mu,\mu)=\Phi(\mu)R_0\Phi^*(\mu)+\int\limits_{\mu_0}^{\mu}\int\limits_{\mu_0}^{\mu}\Phi(\mu)\Phi^{-1}(t)\times \times R_{\psi\psi}(t,\tau)\left[\Phi^{-1}(\tau)\right]^*\Phi^*(\mu)\,dt\,d\tau$$

or in the form

$$R_{YY}(\mu,\mu)=\Phi(\mu)\gamma(\mu)\Phi^*(\mu), \tag{5.51}$$

where

$$\gamma(\mu)=R_0+\int\limits_{\mu_0}^{\mu}\int\limits_{\mu_0}^{\mu}\Phi^{-1}(t)R_{\psi\psi}(t,\tau)\left[\Phi^{-1}(\tau)\right]^*dt\,d\tau. \tag{5.52}$$

Differentiating Eq. (5.51), we get

$$R'_{YY}(\mu)=\Phi'(\mu)\gamma(\mu)\Phi^*(\mu)+\Phi(\mu)\gamma'(\mu)\Phi^*(\mu)+\Phi(\mu)\gamma(\mu)\Phi'^*(\mu).$$

Since

$$\Phi'=\tilde{A}\Phi,\quad (\Phi')^*=\Phi^*A^*,$$

we will have

$$R'_{YY}(\mu)=\tilde{A}R_{YY}+R_{YY}A^*+\Phi(\mu)\gamma'(\mu)\Phi^*(\mu). \tag{5.53}$$

Introducing the notation

$$\varphi(\tau,\mu)=\int\limits_{\mu_0}^{\mu}R_{\psi\psi}(\tau,\lambda)\left[\Phi^{-1}(\lambda)\right]^*d\lambda,$$

let us represent Eq. (5.52) in the following form:

$$\gamma(\mu)=R_0+\int\limits_{\mu_0}^{\mu}\Phi^{-1}(\tau)\varphi(\tau,\mu)\,d\tau$$

and let us differentiate it in the independent variable

$$\gamma'(\mu)=\int\limits_{\mu_0}^{\mu}\Phi^{-1}(\tau)\frac{\partial}{\partial\mu}\varphi(\tau,\mu)\,d\tau+\Phi^{-1}(\mu)\varphi\quad(r=\mu,\mu).$$

Since /175

$$\frac{\partial\varphi(\tau,\mu)}{\partial\mu}=\frac{\partial}{\partial\mu}\int\limits_{\mu_0}^{\mu}R_{\psi\psi}(\tau,\lambda)\left[\Phi^{-1}(\lambda)\right]^*d\lambda=R_{\psi\psi}(\tau,\mu)\left[\Phi^{-1}(\mu)\right]^*,$$

the expression for the derivative of function $\gamma(\mu)$ is of the form:

$$\gamma'(\mu)=\int_{\mu_0}^{\mu}\Phi^{-1}(\tau)R_{\psi\psi}(\tau,\mu)\,d\tau\left[\Phi^{-1}(\mu)\right]^*+$$
$$+\Phi^{-1}(\mu)\int_{\mu_0}^{\mu}R_{\psi\psi}(\mu,\lambda)\left[\Phi^{-1}(\lambda)\right]^*d\lambda.$$

Substituting this equality into Eq. (5.53), we will have the following integro-differential equation:

$$R'_{YY}(\mu)=\tilde{A}R_{YY}+R_{YY}\tilde{A}^*+\Phi(\mu)\int_{\mu_0}^{\mu}\Phi^{-1}(\tau)R_{\psi\psi}(\tau,\mu)\,d\tau+$$
$$+\int_{\mu_0}^{\mu}R_{\psi\psi}(\mu,\tau)\left[\Phi^{-1}(\tau)\right]^*d\tau\,\Phi^*(\mu). \tag{5.54}$$

We introduce the notation

$$\nu(\mu)=\int_{\mu_0}^{\mu}\Phi(\mu)\Phi^{-1}(\tau)R_{\psi\psi}(\tau,\mu)\,d\tau. \tag{5.55}$$

Given the condition that

$$\nu^*(\mu)=\int_{\mu_0}^{\mu}R^*_{\psi\psi}(\tau,\mu)\left[\Phi^{-1}(\tau)\right]^*d\tau\,\Phi^*(\mu),$$

Eq. (5.54) takes on the following form:

$$R'_{YY}=\tilde{A}R_{YY}+R_{YY}\tilde{A}^*+\nu(\mu)+\nu^*(\mu). \tag{5.56}$$

The quantity $\gamma(\mu)$ is a matrix of order (n, n).

Differentiating Eq. (5.55) with respect to parameter of μ, we will have

$$\nu'(\mu)=\tilde{A}\nu(\mu)+R_{\psi\psi}(\mu)+\nu_1(\mu),$$

where

$$\nu_1(\mu)=\Phi(\mu)\int_{\mu_0}^{\mu}\Phi^{-1}(\tau)\frac{\partial}{\partial\mu}R_{\psi\psi}(\tau,\mu)\,d\tau. \tag{5.57}$$

Continuing this process of transformations, we get an infinite system of equations characterizing the correlational transformations: /176

$$R'_{YY} = \tilde{A}R_{YY} + R_{YY}\tilde{A}^* + \nu(\mu) + \nu^*(\mu);$$
$$\nu'(\mu) = \tilde{A}(\mu)\nu(\mu) + R_{\psi\psi}(\mu) + \nu_1(\mu);$$
$$\nu'_1(\mu) = \tilde{A}(\mu)\nu_1(\mu) + R_{\psi\psi'}(\mu) + \nu_2(\mu);$$
$$\nu'_2(\mu) = \tilde{A}(\mu)\nu_2(\mu) + R_{\psi\psi''}(\mu) + \nu_3(\mu);$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\nu'_k(\mu) = \tilde{A}(\mu)\nu_k(\mu) + R_{\psi\psi^{(k)}} + \nu_{k+1}; \qquad (5.58)$$

where

$$\nu_{k+1}(\mu) = \int_{\mu_0}^{\mu} \Phi(\mu)\Phi^{-1}(\tau) M\left[\psi(\tau)\frac{d^{(k)}}{d\mu^k}\psi(\mu)\right]d\tau.$$

By form, the systems of differential equations (5.49) and (5.58) coincide, therefore we can write

$$R_{Y\psi(k+1)}(\mu) = \nu_{(k+1)} = \int_{\mu_0}^{\mu} \Phi(\mu)\Phi^{-1}(\tau) M\left[\psi(\tau)\frac{d^{(k)}}{d\mu^k}\psi(\mu)\right]d\tau. \qquad (5.59)$$

Eq. (5.59) enables us to estimate the magnitude of matrix $R_{Y\psi(k+1)}$, if we use the data on the numerical characteristics of the matrix

$$M\left[\psi(\tau)\frac{d^{(k)}}{d\mu^k}\psi(\mu)\right] = R_{\psi\psi^{(k)}}(\tau, \mu).$$

Let us now determine expressions for $M[\psi(\mu)\psi^{(k)}(\mu)]$. To do this, let us find the derivative of the product

$$\frac{d}{d\mu}\psi(\mu)\psi^*(\mu) = \psi'(\mu)\psi^*(\mu) + \psi(\mu)\psi'^*(\mu). \qquad (5.60)$$

We use the operation of mathematical expectation on the equality (5.60), here considering that the operations of mathematical expectationsand differentiation are permutatable. We get

$$R'_{\psi\psi}(\mu) = R_{\psi'\psi^*} + R^*_{\psi'\psi^*}.$$

Obviously,

$$R_{\psi'\psi^*} = \frac{1}{2}R'_{\psi\psi}. \qquad (5.61)$$

Now let us determine the second derivative of the product $\psi\psi^*$: /177

$$\frac{d^2}{d\mu}\psi\psi^* = \psi''\psi^* + 2\psi'\psi' + \psi\psi''.$$

After the averaging operation, we will have

$$R''_{\psi\psi*} = M[\psi''\psi^*[+ M[\psi\psi''^*] + 2M[\psi'\psi'^*].$$

Since

$$M[\psi'\psi'] = R_{\psi'\psi'} = R''_{\psi\psi*},$$

we get

$$R''_{\psi\psi} = 2M[\psi''\psi^*] + 2R'_{\psi\psi},$$

hence it follows that

$$R_{\psi\psi''} = -\frac{1}{2}R''_{\psi\psi}. \tag{5.62}$$

Continuing these transformations, we can easily obtain

$$R_{\psi\psi'''} = -R'''_{\psi\psi}. \tag{5.63}$$

On analogy, we can find expressions of the form (5.61) - (5.63) also for $R_{\psi\psi(k)}$.

The statistical analysis of fluctuations in atmospheric thermodynamic characteristics enabled us to make an estimate of the variables $R_{\psi\psi}$, $R_{\psi\psi'}$, $R_{\psi\psi''}$, and so on.

It turned out that

$$R_{\psi\psi''} \ll R_{\psi\psi'} \ll R_{\psi\psi}.$$

Therefore, for control processes in the atmosphere we can assume that

$$R_{\psi\psi'} \approx R_{\psi\psi''} \approx \ldots \approx R_{\psi\psi(k)} \approx 0.$$

Accordingly, a system of correlational equations (5.58) becomes considerably simplified and takes on the form:

$$R'_{YY}(\mu) = \tilde{A}(\mu)R_{YY}(\mu) + R_{YY}(\mu)\tilde{A}(\mu) + \nu(\mu) + \nu^*(\mu);$$
$$\nu'(\mu) = \tilde{A}(\mu)\nu(\mu) + R_{\psi\psi}(\mu).$$

This is valid also for the system of equations (5.49). Therefore we will have

$$\left.\begin{aligned} R_{YY'}(\mu) &= \tilde{A}(\mu)R_{YY}(\mu) + R_{YY}(\mu)\tilde{A}^*(\mu) + R_{Y\psi}(\mu) + R^*_{Y\psi}(\mu); \\ R'_{Y\psi}(\mu) &= \tilde{A}(\mu)R_{Y\psi}(\mu) + R_{\psi\psi}(\mu); \\ R_{YY}(\mu_0) &= R_0; \quad R_{Y\psi}(\mu_0) = 0. \end{aligned}\right\} \tag{5.64}$$

Therefore, we arrive at fairly simple differential equations which are highly convenient for the statistical analysis of control processes of flight vehicle motion in the atmosphere based on the linear model (5.35). /178

In the particular case when the perturbation is "white noise" with characteristic (5.46), from Eqs. (5.64) there follows the familiar correlation equation

$$R'_{YY}(\mu) = A'(\mu) R_{YY}(\mu) + R_{YY}(\mu) \tilde{A}^*(\mu) + S(\mu).$$

The results of investigation the linear model (5.35) can be used as the first approximation in investigating nonlinear stochastic differential equations (5.1) of the control process.

Example 5.3. For the linear model (5.1.7) of the control process (5.1.5), given the numerical values of the coefficients of the control algorithm (5.1.6) equal to k_1 = -0.000418, k_2 = =-0.000576, k_3 = -0.000633, and k_4 = -0.000295, the numerical integration of the correlation system of equations is carried out:

$$\begin{aligned} \dot{R}_{XX} &= \tilde{A}(t) R_{XX} + R_{XX}\tilde{A}^*(t) + R_{X\xi} + R^*_{X\xi}, \\ \dot{R}_{X\xi} &= \tilde{A}(t) + \tilde{C} R_{\xi\xi} \tilde{C}^*, \end{aligned} \tag{5.3.1}$$

where

$$R_{XX} = \begin{pmatrix} M\left[V_x^2\right] & M[V_x x] & M[V_y V_x] & M[V_x y] \\ M[V_x x] & M[x^2] & M[x V_y] & M[xy] \\ M[V_x V_y] & M[V_y x] & M\left[V_y^2\right] & M[V_y y] \\ M[V_x y] & M[xy] & M[V_y y] & M[y^2] \end{pmatrix},$$

$$R_{X\xi} = \begin{pmatrix} M[V_x \xi_1] & M[V_x \xi_2] & M[V_x \xi_3] \\ M[x\xi_1] & M[x\xi_2] & M[x\xi_3] \\ M[V_y \xi_1] & M[V_y \xi_2] & M[V_y \xi_3] \\ M[y\xi_1] & M[y\xi_2] & M[y\xi_3] \end{pmatrix},$$

$$R_{\xi\xi} = \begin{pmatrix} M\left[\xi_1^2\right] & M[\xi_1\xi_2] & M[\xi_1\xi_3] \\ M[\xi_1\xi_2] & M\left]\xi_2^2\right] & M[\xi_2\xi_3] \\ M[\xi_1\xi_3] & M[\xi_2\xi_3] & M\left[\xi_3^2\right] \end{pmatrix},$$

$$\tilde{A} = A + BK^*,\ R_X(0) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 10^6 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10^6 \end{pmatrix}.$$

Figs. 5.5 - 5.7 give the graphs of the root mean-square values of the coordinates x, y, V_x, and V_y, control α, and normalized values of the reciprocal moments /179

$$r_{14}=\frac{M[V_x y]}{\sigma_{V_x}\sigma_y},\quad r_{13}=\frac{M[V_x V_y]}{\sigma_{V_x}\sigma_{V_y}},$$
$$r_{12}=\frac{M[V_x x]}{\sigma_{V_x}\sigma_x},\quad r_{34}=\frac{M[V_y y]}{\sigma_{V_y}\sigma_y},$$
$$r_{24}=\frac{M[x\,y]}{\sigma_x\sigma_y},\quad r_{23}=\frac{M[V_y x]}{\sigma_x\sigma_{V_y}}.$$

Analysis of the results of the numerical integration of the correlation system of equations gives a fairly complete representation of the control process (5.1.7).

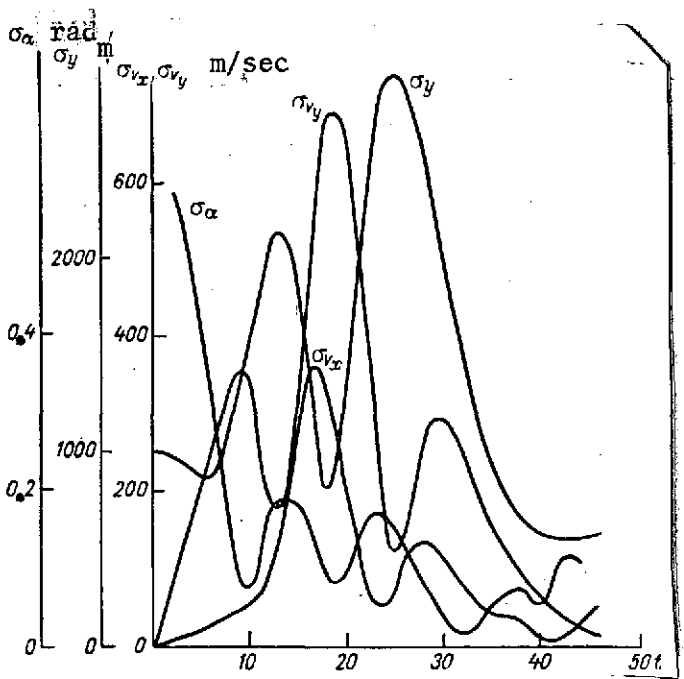


Fig. 5.5. Root mean square values of the coordinates y, V_x, V_y, and control α for the time argument of the program (ϕ = t)

Thus, from Fig. 5.5 we can determine the time intervals when the phase coordinates of the process have their greatest scatter, and also we can estimate at any instant of time within the framework of correlation theory the possible scatter of phase coordinates and of the control.

Fig. 5.6 enables us to establish a correlation between the phase coordinates and shows that a near-functional relationship exists between coordinates x and y. The explanation of this fact lies in the structure of the control algorithm (5.1.6).

Example 5.4. Similar calculations were carried out for the linear parametric model (5.1.7) of the control process. Figs. 5.7 and 5.8 give the graphs of the root mean square values of the coordinates t, y, V_x, V_y, control α, and normalized values of the reciprocal moment r when only the deviations of the initial conditions ΔX_o are in effect.

/180

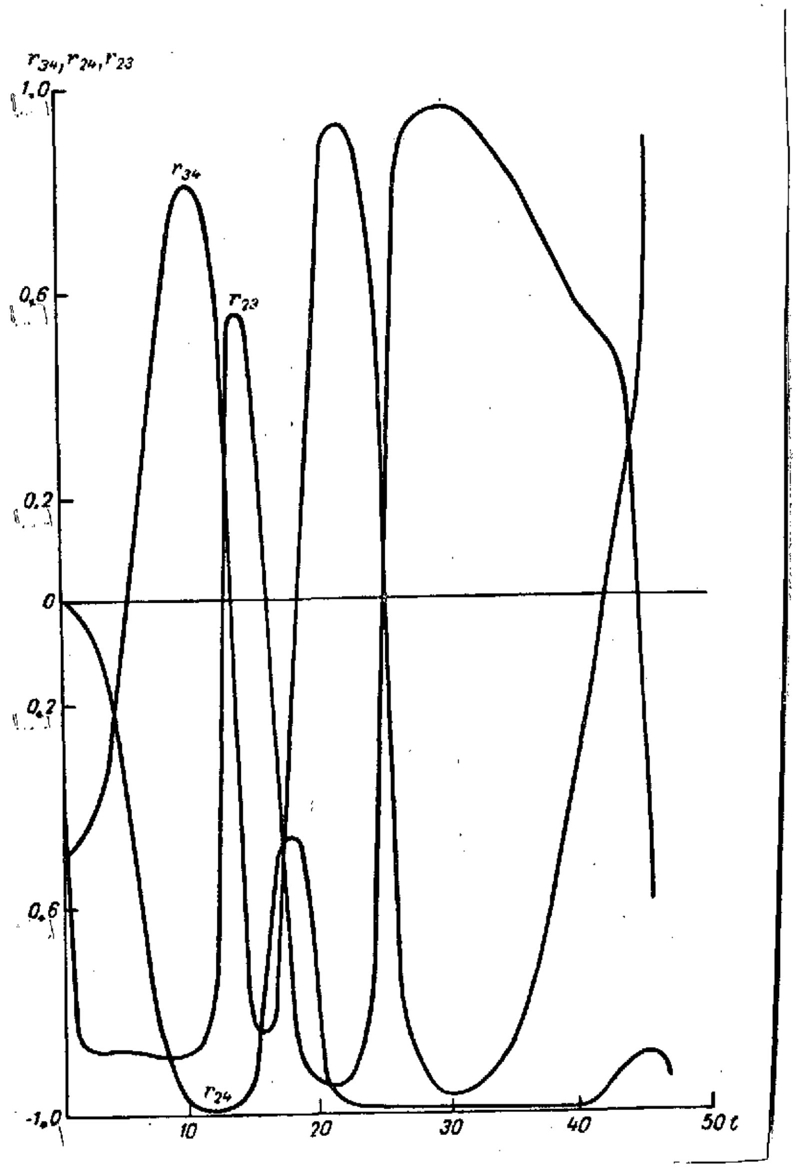


Fig. 5.6. Normalized values of reciprocal moments for the time argument of the program (ϕ = t)

Figs. 5.9 and 5.10 present similar results for zero initial conditions and when only external atmospheric perturbations are in effect. Fig. 5.11 illustrates the results of the effect on the control process of deviations from the initial conditions and external perturbations. /183

A comparison of the results of investigating linear models under the time and parametric form of specifying the argument μ enables us to establish the considerable effect that the form of the argument has on the scatter of phase coordinates of the control

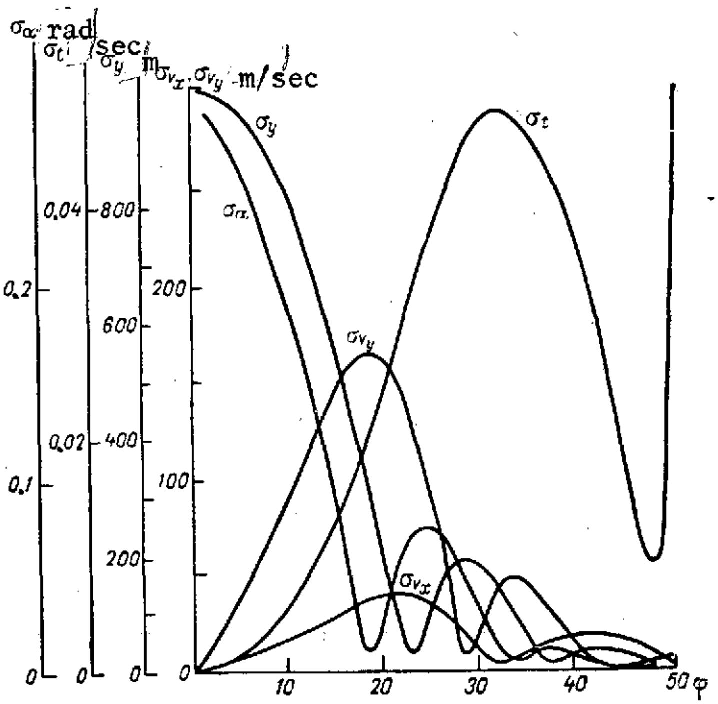


Fig. 5.7. Root mean square values of the coordinates t, V_x, V_y, y, and control α for the parametric argument $\phi = 0.000298x$ caused by scatter of the initial conditions

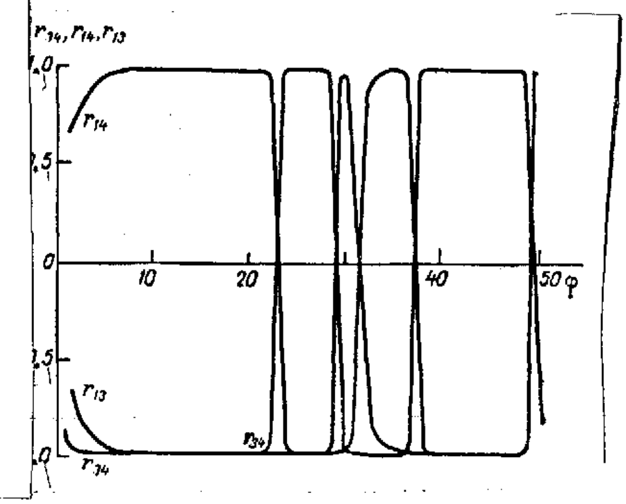


Fig. 5.8. Normalized values of reciprocal moments given the parametric argument $\phi = 0.000298x$ caused by scatter of the initial conditions.

process. Thus, when $\mu = t$, the root mean square value of the coordinate y is $\sigma_y = 3000$ m, while when $\mu = \nu_0 x$, this value is $\sigma_y = 1000$ m. Here, the maximum value of σ_y is shifted with respect to the argument to the beginning of the control process (when $\mu = \nu_0 x$) compared with the time argument. These illustrative examples graphically show the effectiveness of the correlational analysis of nonlinear systems based on the linear model.

5.3. Methods of Statistical Investigation of Nonlinear Processes

When examining methods of the statistical analysis of control processes described by nonlinear stochastic differential equations, a problem of considerable importance is the selection and validation of the corresponding model of the process. Mathematical models of the processes (5.1) and (5.4) are structurally quite close, since /184
in both mathematical models the right side of differential equations

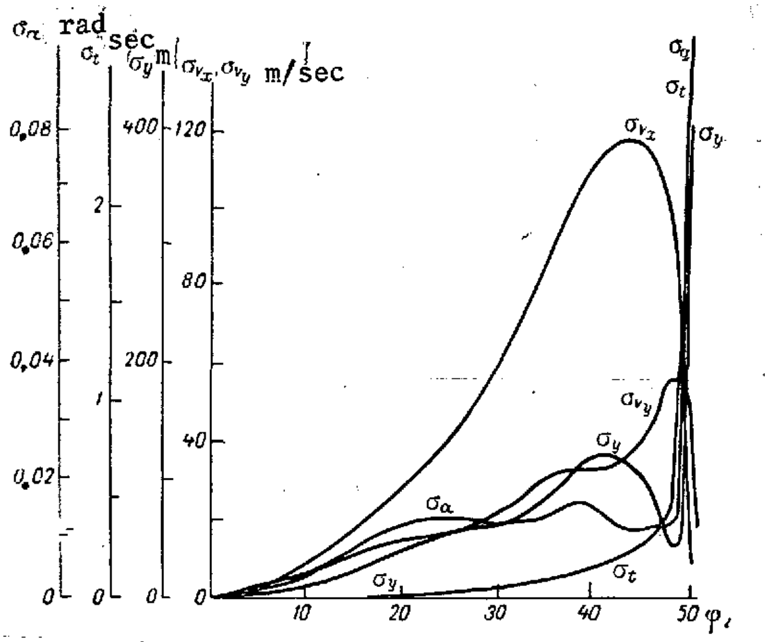


Fig. 5.9. Root mean square values of the coordinates t, y, V_x, V_y, and control α for the parametric argument ϕ = 0.000298x caused by the action of atmospheric perturbations

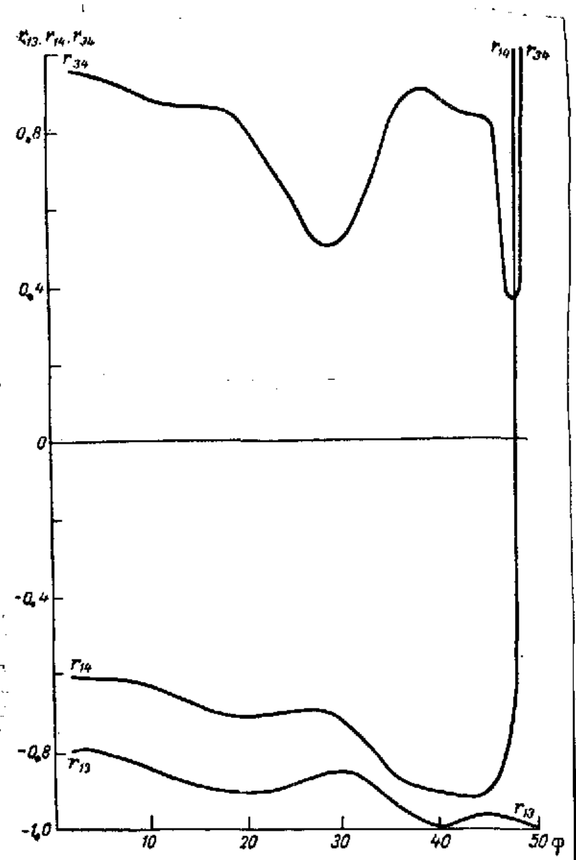


Fig. 5.10. Normalized values of the reciprocal moments for the parametric argument ϕ = = 0.000298x caused by the action of atmospheric perturbations

/182

(5.1) and (5.4) is described by a continuous random process $\xi(t)$ or $\psi(t)$ with assigned statistical characteristics. Naturally, two approaches of investigating processes described by models (5.1) and (5.4) suggest themselves. The first involves correlational transformations, as was done for the linear model of process (5.35), and constructing a system of correlational nonlinear differential equations linking correlation matrices of solutions $R_{XX}(t,t)$, correlational matrices of perturbations $R_{\xi\xi}(t,\tau)$, and the matrix of initial-condition correlations R_o. However, the solution of this problem in particular cases of nonlinear processes involves infinite systems of differential equations and the necessity of computing the moments of solutions of high order $M[X^{(k)}(t)X^{*(p)}(t)]$, where k, p $\geq$ 1 are positive integers. Correlational transformations of nonlinear systems is a quite complicated process and essentially they do not find wide use in the actual practice of investigating nonlinear stochastic processes of the control of flight vehicle motion.

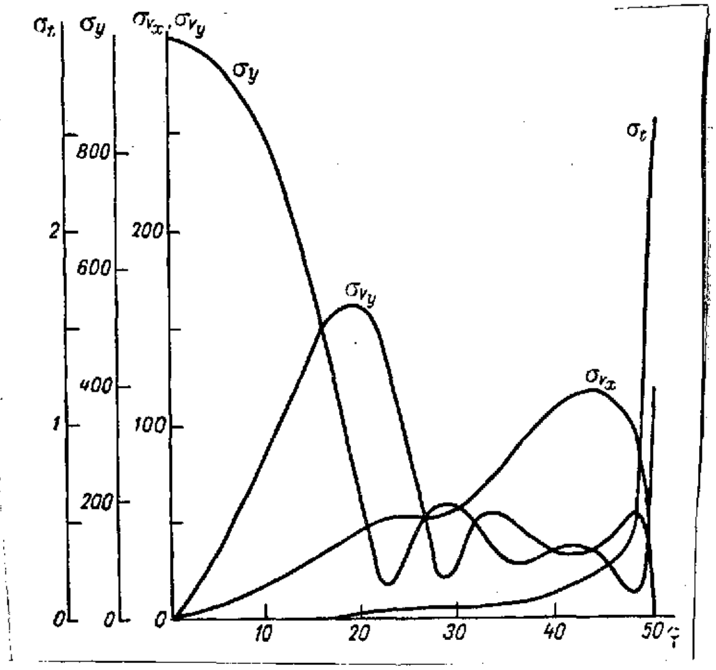


Fig. 5.11. Root mean square values of the phase coordinates t, y, V_x, V_y, and control α for the parametric argument $\phi = 0.000298x$ caused by scatter of the initial conditions and by the action of external perturbations.

The second approach to the statistical investigation of processes described by models (5.1) and (5.4) is associated with approximate methods of computation based on the techniques of mathematical statistics.

In beginning to expound on the problems involved in applyinggthe apparatus of mathematical statistics to the statistical investigations of nonlinear stochastic processes, we introduce a set Ω_z of elements, which we will call sampling or random points. We will regard a sampling point as the possible outcome of an experiment conducted with a given set of conditions (random perturbations). As applied to this particular problem of analysis, by an experiment we will mean the integration of differential equations for specified realizations of random perturbations $\psi(\bar{t})$ or $\xi(t)$ in the mathematical models (5.1) or (5.4). Let us assume that as a result of conducting N experiments (integrations of Eqs. (5.1) employing analog, digital, or hybrid computers) for realization of the random perturbations

$$\xi^{(1)}(t),\ \xi^{(2)}(t),\ \ldots,\ \xi^{(N)}(t) \tag{5.65}$$

we get a set Ω_z of realizations of the solutions to equations (5.1)

$$X^{(1)},\ X^{(2)},\ \ldots,\ X^{(N)}. \tag{5.66}$$

Since the sequence of realizations of perturbations (5.65) is a random sample, the sequence of solutions (5.66) is also a random sample.

From the elements of sample (5.66) let us compose the sample

$$z^{(1)},\ z^{(2)},\ \ldots\ z^{(N)} \tag{5.67}$$

and let us examine several statistics of the sample: /185

the sample mean value

$$\bar{z} = \frac{1}{N}\sum_{i=1}^{N} z^{(i)}; \tag{5.68}$$

the sample dispersion

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(z^{(i)} - \bar{z}\right)^2. \tag{5.69}$$

If the laws of the distribution of the elements of sample (5.67) are identical, and the elements of sample as such are independent random variables, then we know /31/ that

$$M\left[\bar{z}\right] = \mu_z; \tag{5.70}$$

$$\sigma^2\left[\bar{z}\right] = M\left[(\bar{z} - \mu_z)^2\right] = \frac{\sigma_z^2}{N}, \tag{5.71}$$

where σ_z is the root mean square value of the random variable z and μ_z is the mathematical expectation of this variable. Similar formulas can be written out for the sampling dispersion:

$$M\left[s^2\right] = \sigma_z^2; \tag{5.72}$$

$$M\left[(s^2)^2\right] = \frac{\mu_4}{N} + \frac{(N-1)^2 + 2}{N(N-1)}\sigma_z^4; \tag{5.73}$$

$$\sigma^2(s^2) = \frac{1}{N}\left(\mu_4 - \frac{N-3}{N-1}\right)\sigma_z^4. \tag{5.74}$$

Eqs. (5.70) and (5.72) mean that the sample mean $\bar{z}$ and the sample dispersion s^2 are unbiased estimates for the mathematical expectation and the dispersions of random variable z with mathe-mathical expectation μ_z is the dispersion of σ_z^2 and the fourth central moment μ_4. Dispersions of the sample mean z and the sample dispersion can be computed by Eqs. (5.71) and (5.74).

Using Eqs. (5.68) and (5.69), we can compute the mean sample values and the sample dispersion, and by formula

$$s_{ij} = \frac{1}{N-1}\sum_{k=1}^{N}\left(x_{ik} - \bar{x}_i\right)\left(x_{jk} - \bar{x}_j\right) \tag{5.75}$$

also the elements of the sample covariance matrix of solutions to nonlinear differential equations (5.1) represented in the sample (5.66).

In estimating the characteristics of scatter the mean sample value and the sample dispersion obtained by Eqs. (5.71) and (5.74), difficulties arise, since the characteristics of the random variable z, namely σ and μ_4 are not known. If the random variable z has a normal distribution, then Eqs. (5.71) - (5.74) are of the form: /186

$$\sigma^2[\bar{z}] = \frac{\sigma_z^2}{N}; \quad \sigma^2[s^2] = \frac{2\sigma_z^4}{N-1}. \tag{5.76}$$

If we use the sample value of the dispersion s^2 instead of σ^2, then from Eqs. (5.76) there follow the formulas

$$\sigma^2[\bar{z}] = \frac{s^2}{N}; \quad \sigma^2[s^2] = \frac{2(s^2)^2}{N-1}. \tag{5.77}$$

Eqs. (5.77) can be used in estimating the characteristics of the scatter of these statistics or in planning a number of experiments (N) for calculating statistics with specified precision.

The process of the statistical analysis that uses as its basis the theory of the sampling method of mathematical statistics and assuming the construction of sample (5.66) is called the method of statistical tests. This method is quite often used in problems of analyzing the scatter of nonlinear systems.

The method of statistical tests can be used also in analyzing a control process utilizing the mathematical model (5.2). Here the process of obtaining the sample (5.66) is considerably simplified, since constructing the sample of random variables V

$$V^{(1)}, V^{(2)}, \ldots, V^{(N)} \tag{5.78}$$

instead of sampling the realizations of random functions (5.65) to a large extent simplifies the applied aspect of the investigation, since the formation of realizations of the random-variable vector with specified distribution of probability density $f_0(V)$ can be done by using standard programs of random numbers on digital computers.

For the remainder, the procedure of the method of statistical tests for mathematical model (5.2) remains the same as for mathematical model (5.1) and (5.4).

In conclusion, we present formulas for computing estimates of the statistical characteristics of solutions to nonlinear equations (5.1), (5.2), and (5.4) utilizing the method of statistical tests:

$$M[X] \approx \frac{1}{N}\sum_{i=1}^{N} X^{(i)} = (\overline{X})_N;$$
$$M[XX^*] \approx \frac{1}{N-1}\sum_{i=1}^{N} X^{(i)}X^{(i)*} = (\overline{XX^*})_N. \quad (5.79)$$

When computers are used, in place of Eq. (5.79) for interpreting the results of calculations involved in constructing the sequence (5.66), we can organize an ongoing processing if /187 we use recursion relations in computing the estimates $(\overline{X})_N$ and $(\overline{XX^*})_N$ for the statistical characteristics $M[X]$, $M[XX^*]$:

$$(\overline{X})_N = \frac{N-1}{N}(\overline{X})_{N-1} + \frac{1}{N}X^{(N)};$$
$$(\overline{XX^*})_N = \frac{N-2}{N-1}(\overline{XX^*})_{N-1} + \frac{1}{N-1}(XX^*)^{(N)}. \quad (5.80)$$

In Eqs. (5.80), expressions $(\overline{X})_{N-1}$ and $(\overline{XX^*})_{N-1}$ denote the estimates $M[X]$ and $M[XX^*]$ obtained in the preceding experiment involving the method of statistical tests.

In discussing the application of statistical tests to problems in the statistical analysis of nonlinear differential equations, we must consider the question of the convergence of estimates $(\overline{X})_N$, $(\overline{XX^*})_N$ to the real characteristics $M[X]$, $M[XX^*]$ and the precision of computing the sample means $(\overline{X})_N$ and the sample covariance matrix $(\overline{XX^*})_N$. In /79/ are presented working formulas for determining the required volume of the sample (5.67) on the condition that a specified precision of computing the mathematical expectation of the random variable obtained by utilizing the Chebyshev inequality and the Lyapunov limit theorem is achieved.

Table 5.1 gives the values of the number of elements of sample (5.67) as a function of the number ε characterizing the range of error in computing the mathematical expectation of the quantity

$$\delta_N = \frac{\overline{z} - \mu_z}{3\sigma_z}$$

for a specified probability P = 0.99 that inequality $\delta_N \leqslant \varepsilon$ has been satisfied:

TABLE 5.1

ε	0.01	0.05	0.1	0.2
$\overline{N}_1$	100 000	4000	1000	250
$\overline{N}_2$	10 000	400	100	25

In Table 5.1, the number $\overline{N}_1$ is obtained from the Chebyshev inequality, and the number $\overline{N}_2$ is obtained by using the Lyapunov limit theorem. The Chebyshev inequality

$$P[\delta_N \geqslant \varepsilon] \leqslant \frac{1}{9\varepsilon^2 N}$$

gives a very rough estimate of the error of variable δ_N. The estimate obtained by using the Lyapunov limit therum

$$P[\delta_N \leqslant \varepsilon] \approx \Phi\left(3\varepsilon \sqrt{\frac{N}{2}}\right),$$

gives a more exact estimate of the error δ_N.

Therefore, when calculating the volume of the sample (5.67), we must use the second row of Table 5.1. /188

When assigning the volume of sample (5.67), we can also use Eqs. (5.71) and (5.74).

From Table 5.1 it follows that to ensure high precision of computing the estimates $(\overline{X})_N$, $(\overline{XX^*})_N$ utilizing the method of statistical tests requires multiple integration of the system of nonlinear differential equations (5.1) in order to arrive at sample (5.66) of high order. Accordingly, it is necessary to examine other approximate methods of the statistical analysis of nonlinear processes.

This possibility appears only for the model of a control process described by differential equations of the form (5.2), in which the random functions are represented by canonical or noncanonical expansions.

In the method of statistical tests, for the model of process (5.2) there are no hypotheses on the structure of the solutions of an equation in random variables V, and therefore the volume of sample (5.78) of the realizations of vector V, and this means also the volume of the sample of solutions (5.66), for a specified precision of the computations of the statistical characteristics, can be quite large (Table 5.1). Obviously, the use of a priori information on the nature of the relationship between the solutions of equation (5.2) and the elements of the random vector V of the form

$$x_i(t, V) = \varphi(V) \tag{5.81}$$

can significantly cut down the volume of computations in determining the statistical characteristics of function $\Phi(V)$. At the present time several techniques have been developed based on various hypotheses on the nature of the relation (5.81). The principal of these is the method of complete linearization, the method of statistical linearization /19/, the method of partial linearization /62/, the method of incomplete linearization, the B. G. Dostupov method /19/, the interpolational method of V. I. Chernetskiy /79/, the method using the technique of least squares, and the method of statistical nodes.

Let us examine the computational aspects of several of these methods.

The method of complete linearization is based on the hypothesis that the relation (5.81) is linear. The main relations in the method of complete linearization have been examined in Section 5.2. Use of the method of complete linearization requires the direct linearization of the nonlinear equations (5.2). In several cases, direct linearization of nonlinear equations (5.2) is impossible by virtue of the fact that the control ΔU is of a relay type, or for other reasons. Then, by using the hypothesis of the linear dependence of function $\phi(V)$ on random variables V, we can write the following working formula:

$$\varphi(V) = \varphi(V=0) + \sum_{i=1}^{m} \frac{\varphi(v_i^p) - \varphi(V=0)}{v_i^p} v_i, \qquad (5.82)$$

where v_i^p is the realization of element v_i of vector V in the computation of function $\phi(V)$. /189

Eq. (5.82) can in several cases prove preferable to relations in the method of direct linearization of equations (5.2), since here there is no need for a direct linearization of nonlinear equations (5.2).

Introducing the notation

$$b_i = \frac{\varphi(v_i^p) - \varphi(V=0)}{v_i^p}$$

and using Eq. (5.82), let us calculate working relations for determining the mathematical expectations and the dispersion of function $\Phi(V)$:

$$\begin{aligned} M[\varphi(V)] &= \varphi(V=0); \\ \sigma^2[\varphi(V)] &= \sum_{i=1}^{m} b_i^2 \sigma^2[v_i]. \end{aligned} \qquad (5.83)$$

Eqs. (5.83) are written on the assumption that there is no correlation between the elements of vector V and these formulas are quite simple.

Method of incomplete linearization. It can turn out that the function (V) is associated with the random-factor vector by the function /36/

$$\varphi(V) = \varphi(V = 0) + C^{*}\Lambda + \varphi_1(\omega), \tag{5.84}$$

where ω is the vector of strongly varying random factors V not admitting of linearization of equations (5.2); and Λ is a vector of weakly changing random factors V admitting of linearization of equations (5.2).

Naturally, here the sum of the orders of vectors ω and Λ is equal to the order of vector V.

Using relation (5.84), we can easily obtain working formulas for computing the statistical characteristics of the function ϕ (V):

$$\left.\begin{aligned} M[\varphi(V)] &= \varphi(V = 0) + M[\varphi_1(\omega)]; \\ \sigma^2[\varphi(V)] &= C^{*}R_{\Lambda\Lambda}C + M[\varphi_1^2(\omega)] - (M[\varphi_1(\omega)])^2; \\ R_{\Lambda\Lambda} &= M[\Lambda\Lambda^{*}]. \end{aligned}\right\} \tag{5.85}$$

By analyzing the working formulas (5.85), we can note that in the absence of a correlation between elements of vectors ω and Λ, to determine $M[\varphi(V)]$ and $\sigma^2[\varphi(V)]$, we must investigate the nonlinear stochastic system subjected to the random factor ω, whose order is less than the order of the complete vector V, and we must investigate the linear system perturbed by vector Λ, which can significantly reduce the volume of calculations when statistical characteristics of the function ϕ (V) are being determined.

Method of partial linearization. To set forth the method of partial linearization, we can represent the solution ϕ (V) as a Taylor series /190

$$\varphi(V) = \varphi(\omega, \Lambda) = \varphi(\omega, \Lambda = 0) + \sum_{i=1}^{m} \left.\frac{\partial\varphi(\omega, \Lambda)}{\partial\lambda_i}\right|_{\Lambda=0} \lambda_i, \tag{5.86}$$

where, as earlier, Λ is a vector of weakly-varying random variables admitting of linearization.

In Eq. (5.86) $\left.\frac{\partial\varphi}{\partial\lambda_i}\right|$ denotes the partial derivative of the function $\phi(V)$ in elements of vector Λ, which is a random function of vector ω of strongly-varying factors or a function of first-order sensitivity. Relation (5.86) enables us to compute the statistical characteristics of $M[\varphi(V)]$ and $\sigma^2[\varphi(V)]$ based on the formula

$$M[\varphi(V)] = M[\varphi(\omega)],$$
$$\sigma^2[\varphi(V)] = M[\varphi^2(\omega)] + \sum_{i=1}^{m_1} \sigma^2_{\lambda_i} M\left[\left(\frac{\partial\varphi(\omega)}{\partial\lambda_i}\right)\right]^2 - (M[\varphi(\omega)])^2. \quad (5.87)$$

When Eqs. (5.87) are used as working formulas, we must investigate the enlarged system of equations including, on the one hand, equations of the form (5.2), and on the other hand, differential equations of sensitivity of the form [62]:

$$\dot{h}_i = \left.\frac{\partial F}{\partial X}\right|_{\Lambda=0} h_i + \left.\frac{\partial F}{\partial \lambda_i}\right|_{\Lambda=0},$$
$$h_i(t_0) = 0, \quad (i = 1, 2, \ldots, m). \quad (5.88)$$

In Eq. (5.88), we let $h_i = \left.\frac{\partial X}{\partial \lambda_i}\right|_{\lambda=0}$.

The joint integration of Eqs. (5.2) and (5.88) enabled us to calculate the required statistical characteristics of solutions of the initial system (5.2) within the frame of reference of correlation theory, however, here we must know the structure of the function $\phi(V)$ in the form (5.86).

Method of B. G. Dostupov. Underlying the Dostupov method is the hypothesis that the solution of the system of nonlinear equations (5.2) can be sufficiently exactly represented in the form of a segment of a Maclaurin series. We present the exposition for the example of computing the statistical characteristics of the function ϕ with a single random variable v. We have [19]

$$\varphi(v) = \sum_{k=0}^{q} \frac{1}{k!}\left(\frac{\partial^k \varphi}{\partial v^k}\right)_{v=0} v^k. \quad (5.89)$$

Using expansion (5.89), we get working formulas for computing the moments of function ϕ in terms of the realizations $\phi_{(i)}$ obtained for specified samples of random variable v. Let $v_{(1)}$, $v_{(2)}$, ..., $v_{(N)}$ be some sample of values of the random variable v. /191

Then, obviously, the following expression is valid

$$\varphi(v_s) = \sum_{k=0}^{q} \frac{1}{k!}\left(\frac{\partial^k \varphi}{\partial v^k}\right)_{v=0} v_s^k. \quad (5.90)$$

Multiplying the left and right sides of equality (5.90) by some weighting coefficients α_s, let us sum the resulting equations in the subscript s. We will have

$$\sum_{s=1}^{N} \alpha_s \varphi(v_s) = \sum_{k=0}^{q} \frac{1}{k!} \left(\frac{\partial^k \varphi}{\partial v^k}\right)_{v=0} \sum_{s=1}^{N} \alpha_s v_s^k. \tag{5.91}$$

If we assume that

$$\sum_{s=1}^{N} \alpha_s v_s^k = M[v^k] \quad (k = 0, 1, 2, \ldots, q), \tag{5.92}$$

then the identity

$$M[\varphi] = \sum_{s=1}^{N} \varphi(v_s) \alpha_s, \tag{5.93}$$

is valid, since by virtue of (5.89) we have

$$M[\varphi(v)] \approx \sum_{k=0}^{q} \frac{1}{k!} \left(\frac{\partial^k \varphi}{\partial v^k}\right)_{v=0} M[v^k].$$

The system of algebraic equations (5.92) can be used to determine the numerical values of the weighting coefficients α_s and for the realizations of the random variable v_s for which we must calculate the function $\phi(v)$.

The working formula for computing the mathematical expectation of the function $\phi(v)$ is quite simple: it is necessary to add realizations with assigned weight α_s.

In the case of the m-dimensional vector of random variables, relations can be written out that are analogous to Eqs. (5.92) and (5.93) for computing the mathematical expectation of the assigned function $\phi(V)$. We will have

$$\begin{aligned}
\varphi(V^s) &= \sum_{k=0}^{q} \frac{1}{k!} \sum_{r_1=1}^{m} \cdots \sum_{r_k=1}^{m} \left(\frac{\partial^{\Sigma r_k} \varphi}{\partial v_{r_1} \partial v_{r_2} \ldots \partial v_{r_k}}\right)_0 v_s^{r_1} v_s^{r_2} \ldots v_s^{r_k}, \\
\sum_{s=1}^{N} \varphi(V^s) \alpha_s &= \sum_{k=0}^{q} \frac{1}{k!} \sum_{r_1=1}^{m} \cdots \sum_{r_k=1}^{m} \left(\frac{\partial^{\Sigma r_k} \varphi}{\partial v_{r_1} \partial v_{r_2} \ldots \partial v_{r_k}}\right)_0 \sum_{s=1}^{N} \alpha_s v_s^{r_1} v_s^{r_2} \ldots v_s^{r_k}, \\
M[\varphi] &= \sum_{k=0}^{q} \frac{1}{k!} \sum_{r_1=1}^{m} \cdots \sum_{r_k=1}^{m} \left(\frac{\partial^{\Sigma r_k} \varphi}{\partial v^{r_1} \partial v^{r_2} \ldots \partial v^{r_k}}\right)_0 M\left[v^{r_1} v^{r_2} \ldots v^{r_k}\right],
\end{aligned} \tag{5.94}$$

whence we have the relations /192

$$M\left[v^{r_1}v^{r_2}\ldots v^{r_k}\right]=\sum_{s=1}^{N}v_s^{r_1}v_s^{r_2}\ldots v_s^{r_k}\alpha_s\,(r_1,r_2,\ldots,r_k=1,2,\ldots,m); \tag{5.95}$$

$$M[\varphi]=\sum_{s=1}^{N}\alpha_s\varphi\left(v_s^{r_1},v_s^{r_2},\ldots,v_s^{r_k}\right). \tag{5.96}$$

The system of algebraic equations (5.95), like the system of equations (5.92), is used for determining the realization of random variables $V_s^{r_k}$ and the weighting coefficients α_s (s = = 1, 2, ..., N).

Using the above results, we can easily write working formulas for several particular cases /19/.

I. When q = 2, we can easily obtain for uncorrelated and centered elements of random vector V:

$$\begin{gathered}\sum_{s=1}^{N}\alpha_s=1;\quad\sum_{s=1}^{N}\alpha_s v_s^{r_1}=0;\\ \sum_{s=1}^{N}\alpha_s v_s^{r_1}v_s^{r_2}=\begin{cases}0, & r_1\neq r_2\\ \sigma_r^2 & r_1=r_2\end{cases}\quad(r_1,r_2=1,2,\ldots,m).\end{gathered} \tag{5.97}$$

The number of equations in system (5.97) is determined from the formula $M=\frac{(m+1)(m+2)}{2}$. When N = m + 2, we will have the following working formulas:

$$\begin{gathered}M[\varphi(v)]\approx\sum_{s=1}^{m+2}\alpha_s\varphi_s=\frac{1}{m}\left(\sum_{s=1}^{m}\varphi_s(v_s)+\frac{\varphi_{m+2}-\varphi_{m+1}}{2}\right),\\ \sigma^2[\varphi(V)]\approx\frac{1}{m}\left(\sum_{s=1}^{m}\varphi_s^2+\frac{\varphi_{m+2}^2-\varphi_{m+1}^2}{2}\right)-(M[\varphi])^2,\\ v_s=\sigma_s\sqrt{m};\ \varphi_{m+1}=\varphi(v_1,v_2,\ldots,v_m);\\ \varphi_{m+2}=\varphi(-v_1,-v_2,\ldots,-v_m).\end{gathered} \tag{5.98}$$

II. When q = 3, working formulas (5.98) becomes

$$\begin{gathered}M[\varphi]\approx\frac{1}{2m}\sum_{s=1}^{m}(\varphi(+v_s)+\varphi(-v_s));\\ \sigma^2[\varphi(V)]\approx\frac{1}{2m}\sum_{s=1}^{m}(\varphi^2(v_s)+\varphi^2(-v_s))-(M[\varphi])^2,\end{gathered} \tag{5.99}$$

where

$$v_s=\sigma_s\sqrt{m}.$$

Similar formulas for $q > 3$ are presented in /19/. Thus, application of the B. G. Dostupov method boils down to the following: /193

calculating the table of realizations of random variables for which it is necessary to calculate the function ϕ; and

treatment of the calculation results by formulas of the form (5.96) - (5.99).

All the foregoing dealing with computing the mathematical expectation and the dispersion of function ϕ is valid also concerning the elements of vector X(t, V).

Interpolation method of V. I. Chernetskiy. Since the solutions to equations (5.2) are determinate functions of time and the random variables v_j ($j = 1, 2, \ldots, m$), the approximate representation in the form of interpolational polynomials in the factors $v_1, v_2, \ldots, v_m$ is possible for them. Let us denote the realization of element v_i of vector V by the set $\{\Omega_V\}$ in terms of $\{v_{ik_i}\}$, where subscript k_i denotes the competing realization of the element v_i, and let us consider q_i realizations of each element v_i. Obtaining the set of numbers $\{v_{1k_1}, v_{2k_2}, \ldots, v_{mk_m}\}$, let us compute for each element in this set the function $\phi(V)$ for the solutions to equations (5.2), using here the methods of analytic (if this is possible) or the numerical solution of the nonlinear differential equations. We will have

$$\varphi_{k_1, k_2, \ldots, k_m} = \varphi\left(v_{1_{k_1}}, v_{2_{k_2}}, \ldots, v_{m_{k_m}}\right).$$

The integral polynomials, represented approximately by the function ϕ when the method of point interpolation is used, will have the following form:

$$\varphi = \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi_{k_1, k_2, \ldots, k_m} \prod_{j=1}^{m} \frac{\omega_{q_j}(v_j)}{\omega'_{q_j}\left(v_{j_{k_j}}\right)\left(v_j - v_{j_{k_j}}\right)}, \qquad (5.100)$$

where

$$\omega_{q_j}(v_j) = \left(v_j - v_{j_1}\right)\left(v_j - v_{j_2}\right) \ldots \left(v_j - v_{j_{q_j}}\right) \qquad (5.100')$$

is a polynomial of degree q_j with respect to the random variables v_j,

ω'_{q_j} are values of the derivative of polynomial (5.100') in the random variable v_j computed at the point $\{v_{jk_j}\}$.

Eq. (5.100) ensures at the integration nodes the coincidence of the interpolational polynomial and the realizations of random function $\phi(V)$. Let us apply the operation of mathematical expectation to the left and right sides of Eq. (5.100). We will have

$$M[\varphi(V)] = \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi_{k_1, k_2, \ldots, k_m} \rho_{k_1, k_2, \ldots, k_m}. \tag{5.101}$$

The quantities /194

$$\rho_{k_1, k_2, \ldots, k_m} = M\left[\prod_{j=1}^{m} \frac{\omega_{q_j}(v_j)}{\omega'_{q_j}\left(v_{j_{k_j}}\right)\left(v_j - v_{j_{k_j}}\right)}\right] =$$

$$= \int_{-\tilde{v}_1}^{\tilde{v}_1} \cdots \int_{-\tilde{v}_m}^{\tilde{v}_m} f_0(v_1, v_2, \ldots, v_m) \prod \frac{\omega_{q_j}(v_j)\, dv_1\, dv_2 \ldots dv_m}{\omega'_{q_j}\left(v_{j_{k_j}}\right)\left(v_j - v_{j_{k_j}}\right)} \tag{5.102}$$

depend on the sampling of random variables $\left[\rho_{k_1, k_2, \ldots, k_m} = \rho\left(v_{1_{k_1}}, v_{2_{k_2}}, \ldots, v_{m_{k_m}}\right), (k_1 = 1, 2, \ldots, q_1;\ k_2 = 1, 2, \ldots, q_2;\ \ldots;\ k_m = 1, 2, \ldots,\right.$ and are called /79/ Christoffel numbers. By virtue of the earlier assumption that there is no correlation between the elements of random vector V, Eq. (5.102) can be represented as

$$\rho_{k_1, k_2, \ldots, k_m} = \prod_{j=1}^{m} \rho_{k_j}, \tag{5.103}$$

that is, for the vector of independent random variables the Christoffel numbers can be represented as the product of the corresponding one-dimensional Christoffel numbers.

With reference to Eq. (5.103), let us write out the working formulas for computing the mathematical expectation of the function

$$M[\varphi] = \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi_{k_1, k_2, \ldots, k_m} \prod_{j=1}^{m} \rho_{k_j}. \tag{5.104}$$

In /79/ it is shown that the approximations converge with the use of the interpolational method to the exact value of the stochastic characteristic of function ϕ, and in this work it is shown that in this case the Christoffel numbers are the roots of the orthogonal polynomials in a weight that is equal to the density of the distribution of random variable v_i. Here the following theorem /79/ is valid: "If one selects as the interpolation nodes the roots of orthogonal polynomials in a weight that is equal to the density of the distribution of random variable v_i, when n interpolation nodes are used, the interpolation method gives exact values in the class of polynomials of all degrees up to the degree $q = 2n - 1$, inclusively. Here it is not possible to enlarge the class of absolute precision for any other polynomial approximation." Also given in /79/ are the interpolation nodes and the Christoffel numbers for the normal distribution of probabilities.

Thus, selecting the number of nodes for each of the elements of vector V, we can write out the sets of coordinate (nodes) and Christoffel numbers which enable us to determine the entire set of interpolation nodes. For the remainder, it is required to compute the function $\phi(V)$ at these nodes and to treat the results /195
using working formula (5.104). Eq. (5.104) can also be used for computing the mathematical expectation of the square of function $\phi(V)$. To compute the mathematical expectations of the square of function $\phi(V)$, let us use the formula

$$M[\varphi^2] = M\left[\sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi_{k2, k1, \ldots, k_m} \prod_{j=1}^{m} \frac{\omega_{q_j}(v)_j}{\omega'_{q_j}\left(v_{j_{k_j}}\right)\left(v_j - v_{j_{k_j}}\right)} \times \right.$$
$$\left. \times \sum_{\nu_1, \nu_2, \ldots, \nu_m}^{q_1, q_2, \ldots, q_m} \varphi_{\nu_1, \nu_2, \ldots, \nu_m} \prod_{l=1}^{m} \frac{\omega_{q_l}(v_l)}{\omega'_{q_l}\left(v_{l_{\nu_l}}\right)\left(v_l - v_{l_{\nu_l}}\right)}\right] =$$
$$= \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \sum_{\nu_1, \nu_2, \ldots, \nu_m}^{q_1, q_2, \ldots, q_m} \varphi_{k_1, k_2, \ldots, k_m} \varphi_{\nu_1, \nu_2, \ldots, \nu_m} P_{k_1, k_2, \ldots, k_m; \nu_1, \nu_2, \ldots, \nu_m},$$

where

$$P_{k_1, k_2, \ldots, k_m; \nu_1, \nu_2, \ldots, \nu_m} = M\left[\prod_{j=1}^{m} \frac{\omega_{q_j}(v_j)}{\omega'_{q_j}\left(v_{jk_j}\right)\left(v_j - v_{jk_j}\right)} \times \right.$$
$$\left. \times \prod_{l=1}^{m} \frac{\omega_{q_l}(v_l)}{\omega'_{q_l}\left(v_{l_{\nu_l}}\right)\left(v_l - v_{l_{\nu_l}}\right)}\right].$$

For the case of uncorrelated elements of vector V we obviously have

$$P_{k_1, k_2, \ldots, k_m; \nu_1, \nu_2, \ldots, \nu_m} = P_{k_1, k_2, \ldots, k_m} = \prod_{j=1}^{m} P_{kj}, \tag{5.105}$$

where

$$P_{kj} = M\left[\left(\frac{\omega_{q_j}(v_j)}{\omega'_{q_j}\left(v_{j_{k_j}}\right)\left(v_j - v_{j_{k_j}}\right)}\right)^2\right], \tag{5.106}$$

therefore

$$M[\varphi^2(V)] = \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi^2_{k_1, k_2, \ldots, k_m} \sum_{j=1}^{m} P_{kj}.$$

Referring to the property of Christoffel numbers /79/, we will have

$$P_{kj} = \rho_{k_j},$$

and with reference to this Eq. (5.107) /sic, probably Eq. (5.104) is meant/ becomes /196

$$M[\varphi^2(V)] = \sum_{k_1, k_2, \ldots, k_m}^{q_1, q_2, \ldots, q_m} \varphi^2_{k_1, k_2, \ldots, k_m} \prod_{j=1}^{m} \rho_{k_j}. \tag{5.107}$$

As a whole, summing up the foregoing concerning the Chernetskiy interpolational method, we should note that the interpolational method has quite broad potentials in the sense of precision in determining the mathematical expectation of function $\phi(V)$. However, the number of nodes here can be appreciable. This corresponds to a considerable volume of calculations in computing the function at the nodes associated with integrating the system of stochastic differential equations (5.2). Therefore the problem of reducing the order of the vector of strongly varying random variables is one of the key problems in multiple analysis of nonlinear stochastic equations of the form (5.2).

As was shown above, in this case we can use the method of incomplete and partial linearization in which the statistical characteristics $M[\varphi(t, \omega)]$, $M[\varphi^2(t, \omega)]$, $M[h_i(t, \omega)h_i^*(t, \omega)]$ $(i=1, 2, \ldots, m_2)$ can be computed by employing the Dostupov method, the method of statistical tests, or the Chernetskiy interpolational method. Obviously, a combination of the method of incomplete or partial linearization with the interpolational method enables us to considerably reduce the volume of computational work in determining the statistical characteristics of nonlinear stochastic systems.

It should be noted that in the method of statistical tests it appears possible in the course of computation to monitor the convergence of the estimates of statistical characteristics even if only from the fact of their variation with increase in the number of tests, while in methods based on any hypothesis on the structure and nature of the relationship (5.81) this monitoring is not available. This means that in the absence of a priori information on the structure of the function (5.81), it does not appear possible to obtain reliable information on the statistical characteristics of the function $\phi(V)$ employing these methods (the methods of linearization, the Dostupov method, and the interpolational method). Therefore the use of these methods in single statistical analysis of nonlinear processes of the form (5.2) can scarcely be justified if we do not know the structure of the relationship (5.81).

Use of methods of the statistical analysis of nonlinear processes (5.2) based on hypotheses on the structure of the relationship (5.81) is advantageous in multiple analysis of stochastic processes of the control of flight vehicle motion in the earth's atmosphere. However, an additional problem arises here -- the problem of studying the structure of the relationship (5.81) for a substantiated application of one of the methods of the statistical analysis of the control processes. Essentially, this is a problem of resolving the vector of random variables V into components that include weakly-changing and strongly-changing random factors, and determining the components of vector V that significantly or insignificantly affect the control process (5.2). /197

Since this problem is of independent interest, approaches and methods of solving it will be examined in the next chapter.

Of interest is a group of methods of the statistical analysis of nonlinear systems based on using an approximate representation of the function $\phi(V)$ with the polynomial

$$\hat{\varphi}_1(V) = a_0 + \sum_{i=1}^{m} a_i v_i + \sum_{i \leqslant j \leqslant 1}^{m} a_{ij} v_i v_j + \ldots \tag{5.108}$$

on a set Ω_V of random variables V, or its expansion in a Taylor series in the neighborhood of the value V = 0:

$$\hat{\varphi}_2(V) = \varphi(V=0) + \sum_{i=1}^{m} \frac{\partial\varphi}{\partial v_i}\Big|_0 v_i + \frac{1}{2} \sum_{i,\, j=1}^{m} \frac{\partial^2\varphi}{\partial v_i \partial v_j}\Big|_0 v_i v_j + \ldots \tag{5.109}$$

If the representations of the function $\phi(V)$ in the form (5.108) or (5.109) are obtained, that is, if the numerical values of coefficients a_0, a_i, a_{ij}, ... and the partial derivatives $\frac{\partial\varphi}{\partial v_i}, \frac{\partial^2\varphi}{\partial v_i \partial v_j}, \ldots,$ we can posit as the basis of the algorithm of the statistical analysis of the process (5.2) the following relations:

$$M[\varphi(V)] \approx M[\hat{\varphi}(V)], \tag{5.110}$$

$$\sigma^2[\varphi(V)] \approx \sigma^2[\hat{\varphi}(V)] \tag{5.111}$$

and so on.

For the case of the quadratic representation of the function $\phi(V)$ with polynomials (5.108) and (5.109), Eqs. (5.110) and (5.111) become:

$$M\left[\hat{\varphi}_1(V)\right] = a_0 + \sum_{i=1}^{m} a_{ii} M\left[v_i^2\right],$$
$$M\left[\hat{\varphi}_1^2(V)\right] = a_0^2 + \sum_{i=1}^{m} a_i^2 M\left[v_i^2\right] + 2a_0 \sum_{i=1}^{m} a_{ii} M\left[v_i^2\right] +$$
$$+ 2 \sum_{i<j=1}^{m} a_{ij}^2 M\left[v_i^2\right] M\left[v_j^2\right] + \sum_{i=1}^{m} a_{ii}^2 M\left[v_i^4\right]$$

(5.112)

and

$$M\left[\hat{\varphi}_2(V)\right] = \varphi(V=0) + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2\varphi}{\partial v_i^2} M\left[v_i^2\right],$$

/198

$$M\left[\hat{\varphi}_2^2(V)\right] = \varphi^2(V=0) + \sum_{i=1}^{m}\left(\frac{\partial\varphi}{\partial v_i}\right)^2 M\left[v_i^2\right] +$$
$$+ \varphi(V=0)\sum_{i=1}^{m} \frac{\partial^2\varphi}{\partial v_i^2} M\left[v_i^2\right] + \frac{1}{4}\sum_{i=1}^{m}\left(\frac{\partial^2\varphi}{\partial v_i^2}\right)^2 M\left[v_i^4\right] +$$
$$+ \frac{1}{4}\sum_{i\neq j=1}^{m}\left(\frac{\partial^2\varphi}{\partial v_i \partial v_j}\right)^2 M\left[v_i^2\right] M\left[v_j^2\right] \qquad (5.113)$$

respectively.

For the linear model of the function

$$\hat{\varphi}_1(V) = a_0 + \sum_{i=1}^{m} a_i v_i$$

or

$$\hat{\varphi}_2(V) = \varphi(V=0) + \sum_{i=1}^{m} \frac{\partial\varphi}{\partial v_i} v_i$$

Eqs. (5.112) and (5.113) are considerably simplified and are of the form

$$M\left[\hat{\varphi}_1(V)\right] = a_0; \quad M\left[\hat{\varphi}_1^2(V)\right] = a_0^2 + \sum_{i=1}^{m} a_i^2 M\left[v_i^2\right] \qquad (5.114)$$

and

$$M\left[\hat{\varphi}_2(V)\right] = \varphi(V=0)$$
$$M\left[\hat{\varphi}_2^2(V)\right] = \varphi^2(V=0) + \sum_{i=1}^{m}\left(\frac{\partial\varphi}{\partial v_i}\right)^2 M\left[v_i^2\right] \qquad (5.115)$$

respectively.

To employ Eqs. (5.112) - (5.115), we must indicate the approaches in computing the coefficients $a_0, a_i, a_{ij}, \varphi(V=0), \frac{\partial\varphi}{\partial v_i}, \frac{\partial^2\varphi}{\partial v_i \partial v_j}$ and so on.

In computing the coefficients a_0, a_i, and a_{ij}, we can use the method of least squares /35/, enabling us to find the unknown coefficients from the condition of the optimum approximation of the function $\hat{\varphi}(V)$ with the polynomial $\hat{\varphi}(V)$ of given degree in the sense of ensuring the minimum value of the criterion of the approximation quality

$$J_N = \sum_{i=1}^{N} \left[\varphi_{(i)}\left[V^{(i)}\right] - \hat{\varphi}_{1(i)}\left[V^{(i)}\right]\right]^2, \tag{5.116}$$

where $\varphi_{(i)}\left[V^{(i)}\right]$ is the value of the function for the specified realization of vector $V = V^{(i)}$; $\hat{\varphi}_{1(i)}\left[V^{(i)}\right]$ is the value of the polynomial for this very same realization of the vector of random variables V.

Introducing the notation /199

$$A^* = (a_0 a_1 a_2 \ldots a_m a_{11} a_{12} \ldots a_{1m} a_{21} a_{22} \ldots a_{mm}),$$
$$\bar{p}_{(i)} = (1 v_1^{(i)} v_2^{(i)} \ldots v_m^{(i)} v_{11}^{2(i)} v_1^{(i)} v_2^{(i)} \ldots v_m^{2(i)} \ldots),$$
$$S^* = (\varphi_{(1)} \varphi_{(2)} \ldots \varphi_{(m)} \ldots \varphi_{(N)}); \quad \bar{P} = (\bar{p}_{(i)}),$$

we can write criterion (5.116) in the form

$$J_N = [S - \bar{P}A]^* [S - \bar{P}A]. \tag{5.117}$$

Determining the partial derivative of the right and left sides of equality (5.117) in vector A and equating it to zero, we get

$$\frac{\partial J_N}{\partial A} = -2(S - \bar{P}A)^* \bar{P} = 0,$$

whence it follows

$$\bar{P}^* \bar{P} A = \bar{P}^* S. \tag{5.118}$$

We can write the solution to equation (5.118) in the form

$$A = (\bar{P}^* \bar{P})^{-1} \bar{P}^* S. \tag{5.119}$$

In the linear model of function $\phi(V)$, the vector A has the dimension $\overline{N}=m+1$. For the quadratic model of the function $\phi(V)$, the order of vector A increases to $\overline{N}=0{,}5(m^2+3m+2)$, where m is the order of vector V.

Accordingly, the number of equations in computing the function $\phi(V)$ must be larger than $\overline{N}$ in order for a matrix that is the reciprocal of the matrix $\overline{P}^*\overline{P}$ to exist.

Below the numerical value of vector A obtained as a result of computations based on Eq. (5.119) will be referred to as the estimate of vector A and will be denoted with $\hat{A}$. If $N>\overline{N}$, Eq. (5.119) can be used in estimating the criterion of the quality (5.117) characterizing the exactness of the representation of function $\phi(V)$ with polynomial $\hat{\phi}(V)$. Let us present several transformations of Eq. (5.117), using Eq. (5.119). We will have

$$\begin{aligned}\hat{J}_N &= [S-\overline{P}A]^*[S-\overline{P}A] = [S-\overline{P}\hat{A}]^*[S-\overline{P}\hat{A}] = \\ &= [S-\overline{P}(\overline{P}^*\overline{P})^{-1}\overline{P}^*S]^*[S-\overline{P}(\overline{P}^*\overline{P})^{-1}\overline{P}^*S] = \\ &= S^*S - S^*\overline{P}(\overline{P}^*\overline{P})^{-1}\overline{P}^*S\end{aligned} \tag{5.120}$$

or

$$\hat{J}_N = S^*S - S^*\overline{P}\hat{A} = S^*\left[S-\overline{P}\hat{A}\right]. \tag{5.121}$$

Eqs. (5.120) or (5.121) can be used in estimating the exactness of the approximation of $\phi(V)$ with polynomial $\hat{\phi}(V)$. /200

The method of least squares for constructing polynomials (5.108) can be successfully used for a small order of the vector of random variables V and for a low degree of the approximating polynomial.

Table 5.2 gives the numerical values of $\overline{N}$ for polynomials of the first and second degrees. From the table it follows that even for an order of vector V equal to 10, constructing the quadratic polynomial for the function $\phi(V)$ poses considerable difficulties of a computational nature.

Table 5.2

Degree of polynomial	m = 1	5	10	15	20	25
1	2	6	11	16	21	26
2	3	21	66	136	231	351

These difficulties are related to the inversion of the matrix $C = \overline{P}^*\overline{P}$ of order $\overline{N} \times \overline{N}$, containing $\overline{M} = \frac{\overline{N}^2 + \overline{N}}{2}$ distinct elements (when $\overline{N} = 66$, we have $\overline{M} = 2211$).

This procedure of computing the statistical characteristics of the function $\phi(V)$ based on its approximate representation has significant advantages over the Dostupov method, the interpolational method, and the method of partial linearization in that here we can determine the numerical value of the error of representing the function $\phi(V)$ with a polynomial $\hat{\phi}(V)$ with a given degree in the form

$$\sigma^2[\varepsilon] \simeq \frac{1}{N-1} \hat{J}_N - (M[\varepsilon])^2,$$

$$M[\varepsilon] \simeq \frac{1}{N} \sum_{l=1}^{N} \varepsilon_{(l)},$$

where

$$\varepsilon_{(i)} = S_{(l)} - \overline{P}_{(l)} \hat{A}.$$

Knowledge of the characteristic σ_ε^2 of error ϵ of the representation of function $\phi(V)$ with polynomial $\hat{\phi}(V)$ enables us to write the expression

$$\varphi(V) = \hat{\varphi}(V) + \varepsilon$$

and to indicate the error in computing the required characteristics of the function $\phi(V)$: /201

$$M[\varphi(V)] = M\left[\hat{\varphi}(V)\right] + M[\varepsilon]; \quad \sigma^2[\varphi(V)] = \sigma^2\left[\hat{\varphi}(V)\right] + \sigma^2[\varepsilon]. \tag{5.122}$$

This method of computing the statistical characteristics of a function based on its approximate representation can be used also in other systems of representing the function $\hat{\phi}(V)$. In principle, we can form an arbitrary system of functions $\gamma_i(v_i, \chi_i)$ and set up a representation of the form

$$\hat{\varphi}(V) = a_0 + \sum_{i=1}^{m} a_i \gamma_i(v_i, \chi_i) + \sum_{i,j=1}^{m} a_{ij} \gamma_i(v_i, \chi_i) \gamma_j(v_j, \chi_j) \tag{5.123}$$

or

$$\hat{\varphi}(V) = a_0 + \sum_{i=1}^{m} a_i \gamma_i(v_i, \chi_i). \tag{5.124}$$

For example, the functions $\gamma_i(v_i)$ can be of the form:

$$\gamma_i(v_i, \chi_i) = \sin \chi_i v_i;$$
$$\gamma_i(v_i, \chi_i, b_i, \nu_i) = \sin \chi_i v_i + b_i \cos \nu_i v_i$$

and so on, where b_i, χ_i, ν_i are the unknown constants.

In this case, to find the estimates of all the unknown coefficients $a_0, a_i, a_{ij}, \ldots, \chi_i, \nu_i, b_i$, and so on, the method of least squares in the above-presented form cannot be employed because the unknown coefficients can appear nonlinearly in the selected system of functions $\gamma_i(v_i)$ and in Eq. (5.123).

Substituting Eqs. (5.124) into criterion (5.116)

$$J_N = \sum_{i=1}^{N} \left[\varphi_{(i)} [V^{(i)}] - a_0 - \sum_{j=1}^{m} a_j \gamma_j (V_j, \chi_j) \right]^2, \tag{5.125}$$

let us set up a function of many variables in the unknown coefficients a_i and χ_i. To find the numerical values of parameters a_i and χ_i, in practice it is difficult to use the necessary conditions for the minimum of criterion (5.125) in the form

$$\frac{\partial J_N}{\partial a_i} = 0; \quad \frac{\partial J_N}{\partial \chi_i} = 0, \tag{5.126}$$

since the system of equations (5.126) is nonlinear relative to a_i and χ_i and its direct solution usually poses considerable difficulties. Here we can find of assistance the numerical methods of seeking the extremum of a function of many variables J_N. This problem is quite simple computationally, since the criteria (5.125) is analytically expressed in terms of the unknown parameters and multiple computation of its numerical value on a digital computer does not usually represent a barrier to the approach of constructing a good approximation of the function $\hat{\phi}(V)$. An advantage of the /202 process of computing the statistical characteristics of the function $\phi(V)$ by its representation $\hat{\phi}(V)$ lies in the fact that for a restricted number of experiments in the computation of the function $\phi(V)$ for specified realizations of a vector of random functions V, we can construct an approximation of the function $\hat{\phi}(V)$, estimate the error of the resulting approximation of the function, and find the required characteristics of the desired function.

This approach of the statistical analysis of control processes in the earth's atmosphere is obviously necessary when computing one realization of the function $\phi(V)$ involves large outlays of computer time in integrating equations (5.2).

When setting up the model (5.109) for the function $\phi(V)$, we must compute the partial derivative $\frac{\partial\varphi}{\partial v_i}, \frac{\partial^2\varphi}{\partial v_i \partial v_j}, \ldots,$ employing the solutions to Eq. (5.2). In practical applications, the function $\phi(V)$ can be given as a function of the phase coordinates of the process (5.2), for example, in the form

$$\varphi_1(V) = G_1(X, T, V) \tag{5.127}$$

or

$$\varphi_2(V) = \int_0^T G_2(X, t, V)\,dt, \tag{5.128}$$

where T is the duration of the control process.

Since the partial derivative $\frac{\partial\varphi}{\partial v_i}, \frac{\partial^2\varphi}{\partial v_i \partial v_j}$ for Eqs. (5.127) and (5.128) are of the form:

$$\begin{gathered}
\frac{\partial\varphi_1}{\partial v_i} = \frac{\partial G_1}{\partial X}\frac{\partial X}{\partial v_i} + \frac{\partial G_1}{\partial v_i};\\
\frac{\partial^2\varphi_1}{\partial v_i \partial v_j} = \left(\frac{\partial X}{\partial v_i}\right)^* \frac{\partial^2 G_1}{\partial X^2}\frac{\partial X}{\partial v_j} + \frac{\partial G_1}{\partial X}\frac{\partial^2 X}{\partial v_i \partial v_j} + \frac{\partial^2 G_1}{\partial v_i \partial v_j}\frac{\partial X}{\partial v_j} + \frac{\partial^2 G_1}{\partial v_i \partial v_j};\\
\frac{\partial\varphi_2}{\partial v_i} = \int_0^T \left[\frac{\partial G_2}{\partial X}\frac{\partial X}{\partial v_i} + \frac{\partial G_2}{\partial v_i}\right] dt;\\
\frac{\partial^2\varphi_2}{\partial v_i \partial v_j} = \int_0^T \left[\left(\frac{\partial X}{\partial v_i}\right)^* \frac{\partial^2 G_2}{\partial X^2}\frac{\partial X}{\partial v_j} + \frac{\partial G_2}{\partial X}\frac{\partial^2 X}{\partial v_i \partial v_j} + \frac{\partial^2 G_2}{\partial v_i \partial X}\frac{\partial X}{\partial v_j} + \frac{\partial^2 G_2}{\partial v_i \partial v_j}\right] dt,
\end{gathered} \tag{5.129}$$

then, introducing the notation

$$h_i = \frac{\partial X}{\partial v_i},\ h_{ij} = \frac{\partial^2 X}{\partial v_i \partial v_j}\ (i \leqslant j = 1, 2, \ldots, m),$$

we will have /203

$$\begin{gathered}
\frac{\partial\varphi_1}{\partial v_i} = \frac{\partial G_1}{\partial X} h_i + \frac{\partial G_1}{\partial v_i},\\
\frac{\partial^2\varphi_1}{\partial v_i \partial v_j} = h_i^* \frac{\partial^2 G_1}{\partial X^2} h_j + \frac{\partial G_1}{\partial X} h_{ij} + \frac{\partial^2 G_1}{\partial v_i \partial v_j} h_j + \frac{\partial^2 G_1}{\partial v_i \partial v_j},\\
\frac{\partial\varphi_2}{\partial v_i} = \int_0^T \left[\frac{\partial G_2}{\partial X} h_i + \frac{\partial G_2}{\partial v_i}\right] dt,\\
\frac{\partial^2\varphi_2}{\partial v_i \partial v_j} = \int_0^T \left[h_i^* \frac{\partial^2 G_2}{\partial X^2} h_j + \frac{\partial G_2}{\partial X} h_{ij} +\right.\\
\left. + \frac{\partial^2 G_2}{\partial v_i \partial X} h_j + \frac{\partial^2 G_2}{\partial v_i \partial v_j}\right] dt\ (i \leqslant j = 1, 2, \ldots, m).
\end{gathered} \tag{5.130}$$

In Eq. (5.130) h_i and h_{ij} are functions of the sensitivity of the solutions to Eq. (5.2) in random variables v_i ($i = 1, 2, \ldots, m$). To compute them, we can use the differential equations of sensitivity /50/. Let us write out the differential equations of sensitivity for a nonlinear model of the control process (5.2).

Obviously, for Eqs. (5.2) represented in the form

$$\left.\begin{aligned} &\dot{x}_l = f_l(x_1, x_2, \ldots, x_m, V, t),\\ &x_l(t_0) = x_{l,0} \ (l = 1, 2, \ldots, n).\end{aligned}\right] \tag{5.131}$$

the following system of differential sensitivity equations of the first and second orders:

$$\left.\begin{aligned} &h_i = \left(h_i^{(1)}, h_i^{(2)}, \ldots, h_i^{(n)}\right),\\ &h_{ij} = \left(h_{ij}^{(1)}, h_{ij}^{(2)}, \ldots, h_{ij}^{(n)}\right),\\ &\dot{h}_i^{(l)} = \sum_{j=1}^{n} \frac{\partial f_l}{\partial x_j} h_i^{(j)} + \frac{\partial f_l}{\partial v_i},\\ &\dot{h}_{ij}^{(l)} = \sum_{k=1}^{n} \frac{\partial f_l}{\partial x_k} h_{ij}^{(k)} + \sum_{k,\nu=1}^{n} \frac{\partial^2 f_l}{\partial x_k \partial x_\nu} h_i^{(k)} h_j^{(\nu)} + \sum_{\nu=1}^{n} \frac{\partial^2 f_l}{\partial v_i \partial x_\nu} h_j^{(\nu)} + \frac{\partial^2 f_l}{\partial v_i \partial v_j},\\ &h_i(t_0) = h_{ij}(t_0) = 0 \quad (i \leqslant j = 1, 2, \ldots, m).\end{aligned}\right] \tag{5.132}$$

The joint integration of systems of equations (5.131) and (5.132) with $V = 0$ enables us during a single integration to compute the function $\phi(V = 0)$ in all the necessary partial derivatives for setting up a model of the function $\hat{\phi}(V)$ in accordance with Eqs. (5.130). /204

It should be noted that model (5.109) can be used only if the function $\phi(V)$ is differentiable with respect to its arguments.

Analysis of the above-presented approximate methods of the statistical analysis of a control process described by Eqs. (5.2) enables us to establish the common ground in approximate methods of the statistical analysis of nonlinear systems:

1. Selection of the realizations of vector V (sequence (5.78)) in the space Ω_V in which the function $\phi(V)$ is computed for the solutions to Eq. (5.2);

2. Treatment of the elements of sequence (5.66) composed of realizations of the function $\phi(V)$ or solutions to Eq. (5.2) for elements of the sequence (5.78); and

3. Verification of the exactness of the resulting estimates of the statistical characteristics.

Setting up the sequence (5.78) of realizations of random vector V in the method of statistical tests, the Dostupov method, and the interpolational method usually involves properties of the density of the probability distribution of vector V. Essentially, in the method of statistical tests, the elements of sequence (5.78) must satisfy the assigned law of the distribution of probabilities $f_o(V)$ of vector V. In the interpolational method, the properties of the function $f_o(V)$ are used in computing both the interpolation nodes as well as the Christoffel numbers. In the Dostupov method, the properties of function $f_o(V)$ are used both in computing the nodes at which the function is computed, as well as in determining the weighting coefficients α_s (s = 1, 2, ...) necessary in solving problem 2, that is, treatment of sequence (5.66). However, the elements of sequence (5.78) in the Dostupov method and in the interpolational method are not associated in explicit form with the density of the distribution of probabilities of vector V.

The solution to the problem of setting up the sequences (5.78) and (5.66) in the methods of statistical analysis of nonlinear systems can be viewed as a process of experiment planning, since experiment planning is the planning of a sequence of tests (experiments) following a scheme that exhibits some optimal properties [77].

Since the selection of sequence (5.78) in these methods is associated with the algorithm of treatment of sequence (5.66), obviously the first problem in methods of the statistical analysis of nonlinear processes is the problem of experiment planning. Accordingly, in the method based on using the models of function $\phi(V)$, computing the sequence (5.78) can be based on methods of the theory of optimal experiment planning [42, 44] and the methods of multifactor analysis.

5.4. Method of Statistical Analysis of Control Processes with Discrete-Continuous Model of Perturbations /205

In the discrete-continuous model of specifying the perturbing actions, the process of investigating the statistical characteristics of a nonlinear stochastic system becomes considerably simplified both as to the planning of experiments as well as in the interpretation of their results. Suppose that in this model of perturbations all the discrete random variables are arranged in the form of the sequence

$$A_1, A_2, \ldots, A_N, \tag{5.133}$$

for which the probabilities of the states of the discrete random variables

$$p_1^{(i_1)}, p_2^{(i_2)}, \ldots, p_N^{(i_N)}. \tag{5.134}$$

are specified.

In the sequence (5.133) we can single out the events

$$A_j = A_j\left\{A_1^{(i_1)}, A_2^{(i_2)}, \ldots, A_N^{(i_N)}\right\}, \quad j = 1, 2, \ldots, \rho_1, \tag{5.135}$$

and the probability for each of these can be computed by using the series (5.134).

Then we can write that the right-hand sides of the nonlinear differential equations of the control process will be determined by the sequence (5.135) of events, that is,

$$\dot{X} = F(x, A_j, t) \quad X(t_0) = X_0. \tag{5.136}$$

Obviously, the solutions to Eq. (5.136) will also be determined only by the series of events (5.135), that is,

$$X(t) = X(t, A_j). \tag{5.137}$$

Thus, the solutions of nonlinear equation (5.136) are discrete random variables with possible values (5.137) with probabilities $\{p[A_j], (j = 1, 2, \ldots, \rho_1)\}$.

To compute the central and initial moments of k-th order for several functions of the solutions to Eq. (5.136), for example, $\phi(X)$, we can employ the following formulas from probability theory:

$$\alpha_k = M[\varphi^k] = \sum_{j=1}^{\rho_1} \varphi^k[A_j]\, p[A_j], \tag{5.138}$$

$$\mu_k = M[(\varphi - M[\varphi])^k] = \sum_{j=1}^{\rho_1}\left(\varphi[A_j] - \sum_{j=1}^{\rho_1}\varphi[A_j]\,p(A_j)\right)^k p[A_j]. \tag{5.139}$$

The resulting working formulas (5.138) and (5.139) are quite /206 simple for computing the statistical characteristics of the assigned functions of solutions to Eq. (5.136). Let us illustrate the foregoing with an example of a stochastic system that has two discrete random variables in the right-hand side, each of which has three states (-1, 0, 1) with probability 1/3. Then the matrix of experiment planning will be of the form

$$A_j = \left\{ \begin{array}{c|c|c|c|c|c|c|c|c} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{array} \right\}.$$

The probability of each of the events A_j is $p[A_j] = \frac{1}{9}$. Obtaining for each event A_j ($j = 1, 2, \ldots, 9$) the solutions to Eq. (5.136), we can compute the statistical characteristics of the function $\phi(X)$ based on Eqs. (5.138) and (5.139), for example,

$$M[\varphi] = \frac{1}{9} \sum_{j=1}^{9} \varphi[A_j],$$

$$M[\varphi^2] = \frac{1}{9} \sum_{j=1}^{9} \varphi^2[A_j]$$

and so on.

The applicability of this method to problems of investigating the scatter of flight vehicle trajectories as well as other stochastic processes will be determined by the number of random factors and by the number of their states, since they define the number of solutions to nonlinear differential equations (5.136).

Accordingly, even for the discrete-continuous model of representation of perturbations, high significance lies in the problem of analyzing the significance of random variables and reducing their number, given the condition that this does not lead to considerable errors in the numerical values of the statistical characteristics of the solutions of the nonlinear equations.

CHAPTER SIX

/207

INVESTIGATING THE EFFECT OF ATMOSPHERIC PERTURBATIONS ON THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS

6.1. Investigating the Significance of Atmospheric Perturbations

The problem of investigating the significance of random perturbations in the motion of flight vehicles bears both independent significance in investigations of the motion of flight vehicles in dense atmospheric layers, as well as applied significance in multiple statistical analysis of the trajectories of flight vehicles in problems of optimizing control systems. The necessity of investigating the significance of random variables v_i ($i = 1, 2, \ldots, m$) of the mathematical model (5.2) in forming the scatter of phase coordinates X with respect to their unperturbed (reference) values $\tilde{X}$ arises when solving problems of the numerical optimization of processes of controlling the motion of flight vehicles in dense atmospheric layers when adopting solutions on the utility of estimating and predicting both atmospheric perturbations as well as aeroballistic parameters of flight vehicles.

If the motion of a flight vehicle is described by differential equation (5.2), and if the quality of the probess is characterized by some function $I=M[\varphi_1(X, T)]$, then by virtue of the dependence of the solutions to Eq. (5.2) on the elements of the random vector V: $x_i(t) = x_i(t, V)$ $(i=1, 2, \ldots, n)$, the function $\varphi_1(X, T)$ in implicit form depends on the random values of vector V and is of the form $\varphi(V)=\varphi_1(X, T)$.

The problem of investigating the significance of random variables v_i in forming the scatter of the function $\phi(V)$ is classified as a problem in multifactor analysis; in recent years, a fairly large number of studies [42, 44, 58, 75, 77, 81] have been devoted to elaborating and applying this class of analysis in various fields of technology. Underlying methods of multifactor analysis of the function $\phi(V)$ is the problem of obtaining some

model of the function $\hat{\phi}(V)$ on the condition of the optimal approximation of the model with respect to the test function $\phi(V)$. Since in practical problems it is impossible to obtain an analytic expression for the function $\phi(V)$ owing to its dependence on the solutions to differential equation (5.2), except for the case of describing the motion of a flight vehicle with linear differential equations /208
of the form (5.10), we have to limit ourselves usually to representing it with the polynomial

$$\hat{\varphi} = a_0 + \sum_{i=1}^{m} a_i v_i + \sum_{i \leq j=1}^{m} a_{ij} v_i v_j + \ldots . \tag{6.1}$$

The problem of constructing an approximating polynomial (6.1) can be solved within the frame of reference of classical regression analysis, employing the method of least squares [1]. To do this, obviously we must form some sequence of the random vector (5.78), and for each of its elements we must compute, by integrating equations (5.2), the function $\phi(V)$, that is, we must set up a sequence of functions (sample of realizations):

$$\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(N)}, \tag{6.2}$$

and then we must treat the sequences (5.78) and (6.2) by employing relations (5.109) and (5.121).

Setting up polynomial (6.1), by analyzing its components on the set Ω we must solve the problem of estimating the contribution of each random variable to forming the scatter of the numerical value of the function $\phi(V)$.

The dispersion [81], component [1], and factor [35] analyses, and methods of experiment planning and interpretation [31, 36, 42, 77] enable us to solve the problem of estimating the contribution made by random factors (elements of vector V) to forming the scatter of function $\phi(V)$.

Let us assume that we have set up a first-degree polynomial

$$\hat{\varphi}(V) = a_0 + \sum_{i=1}^{m} a_i v_i \tag{6.3}$$

and that we have computed the statistical characteristics of the function ϕ:

$$M[\varphi(V)],\ M[\varphi^2(V)],\ \sigma[\varphi(V)].$$

Let us obtain estimates of the first two moments of the function $\hat{\phi}(V)$ by using representation (6.3), on the assumption that there is no correlation between the elements of vector V and the centeredness of its elements. We will have

$$\left.\begin{aligned} M\left[\hat{\varphi}(V)\right] &= a_0, \\ M\left[\hat{\varphi}^2(V)\right] &= a_0^2 + \sum_{i=1}^{m} a_i^2 M\left[v_i^2\right], \\ \sigma^2\left[\hat{\varphi}(V)\right] &= \sum_{i=1}^{m} a_i^2 M\left[v_i^2\right]. \end{aligned}\right] \qquad (6.4)$$

Under the normal and equiprobable $(v_i \in [-b_i, b_i])$ distributions of the elements of random vector V, Eqs. (6.4) become: /209

$$\left.\begin{aligned} M\left[\hat{\varphi}(V)\right] &= a_0, \\ M\left[\hat{\varphi}^2(V)\right] &= a_0^2 + \sum_{i=1}^{m} a_i^2 \sigma_i^2, \\ \sigma^2\left[\hat{\varphi}(V)\right] &= \sum_{i=1}^{m} a_i^2 \sigma_i^2 \end{aligned}\right]$$

and

$$\left.\begin{aligned} M\left[\hat{\varphi}(V)\right] &= a_0, \\ M\left[\hat{\varphi}^2(V)\right] &= a_0^2 + \frac{1}{3}\sum_{i=1}^{m} a_i^2 b_i^2, \\ \sigma^2\left[\hat{\varphi}(V)\right] &= \frac{1}{3}\sum_{i=1}^{m} a_i^2 b_i^2. \end{aligned}\right]$$

From Eq. (6.4) it follows that the contribution of the i-th factor to forming the dispersion of function $\phi(V)$ is defined by the expression

$$\eta_i^{(1)} = a_i^2 M\left[v_i^2\right], \; (i = 1, 2, \ldots, m). \qquad (6.5)$$

Let us introduce the coefficient of significance for the first-degree polynomial

$$\delta\eta_i^1 = \frac{\eta_i^1}{\sigma^2\left[\varphi(V)\right]}. \qquad (6.6)$$

Table 6.1 gives the expressions for Eq. (6.5) and (6.6) for the normal and the uniform /equiprobable/ laws of the distribution of elements of vector V.

Similar relations can be obtained for the polynomial of second degree:

$$\hat{\varphi}(V) = a_0 + \sum_{i=1}^{m} a_i v_i + \sum_{i\leqslant j=1}^{m} a_{ij} v_i v_j. \tag{6.7}$$

Table 6.1

Notation	General formula	Formulas under normal distr.	Formulas under uniform distribution
$\eta_i^{(1)}$	$a_i^2 M[v_i^2]$	$a_i^2\sigma_i^2$	$\frac{1}{3} a_i b_i^2$
$\delta\eta_i^{(1)}$	$\eta_i^{(1)}/\sigma^2[\varphi]$	$a_i^2 \frac{\sigma_i^2}{\sigma^2[\varphi]}$	$\frac{1}{3} a_i^2 \frac{b_i^2}{\sigma^2[\varphi]}$

We will have /210

$$M\left[\hat{\varphi}\right] = a_0 + \sum_{i=1}^{m} a_{ii} M[v_i^2],$$

$$M\left[\hat{\varphi}^2(V)\right] = a_0^2 + 2a_0 \sum_{i=1}^{m} a_{ii} M[v_i^2] + \sum_{i=1}^{m} a_i^2 M[v_i^2] +$$
$$+ \sum_{i=1}^{m} a_{ii}^2 M[v_i^4] + 2 \sum_{i<j=1}^{m} a_{ii} a_{jj} M[v_i^2] M[v_j^2] + \sum_{i<j=1}^{m} a_{ij}^2 \times$$
$$\times M[v_i^2] M[v_j^2],$$

$$\sigma^2\left[\hat{\varphi}(V)\right] = \sum_{i=1}^{m} \{a_i^2 M[v_i^2] + a_{ii}^2 [M[v_i^4] - (M[v_i^2])^2]\} +$$
$$+ \sum_{i<j=1}^{m} a_{ij}^2 M[v_i^2] M[v_j^2].$$

We introduce the notation

$$\bar{a}_{ij} = \bar{a}_{ji} = \frac{1}{2} a_{ij}.$$

Then we get the following expression for the coefficient $\eta_i^{(2)}$, characterizing the contribution of the i-th factor to forming the dispersion of the function $\phi(V)$ with respect to the second-degree polynomial

$$\eta_i^{(2)} = a_i^2 M[v_i^2] + a_{ii}^2 [M[v_i^4] (M[v_i^2])^2] +$$
$$+ M[v_i^2] \sum_{j=1}^{m} \frac{\bar{a}_{ij}^2}{4} M[v_j^2]. \tag{6.8}$$

Let us define the coefficient of significance by the formula

$$\delta\eta_i^{(2)} = \frac{\eta_i^{(2)}}{\sigma^2[\varphi]} .$$

Using the computed coefficients $\delta\eta^{(1)}$ and $\delta\eta^{(2)}$, we can divide all random factors into essential and unessential. Unessential factors need not be taken into account in the further investigations. Of the essential factors, we can distinguish the weakly and strongly varying factors, that is, factors for which we can either use a linear model or set up a more complicated nonlinear mathematical model. To solve this problem, we must compare the linear and nonlinear (quadratic) models of the function under study.

Let us define the coefficient of nonlinearity by using the relation /211

$$\delta\Delta\eta_i = \frac{\Delta\eta_i}{\eta_i^{(2)}}, \tag{6.9}$$

where

$$\Delta\eta_i = \eta_i^{(2)} - \eta_i^{(1)}. \tag{6.10}$$

Substituting (6.10) into Eq. (6.9), we can obtain the following expression for estimating the coefficient of nonlinearity:

$$\delta\Delta\eta_i = 1 - \delta\Delta\bar{\eta}_i, \tag{6.11}$$

where

$$\delta\bar{\eta}_i = \frac{\eta_i^{(1)}}{\eta_i^{(2)}} .$$

Table 6.2 gives a listing of the main working relations for solving the problem of estimating the significance of random factors by employing a second-degree model.

By computing the numerical values of the coefficients $\delta\Delta\bar{\eta}_i$, $\delta\Delta\eta_i, \delta\eta_i^{(1)}, \delta\eta_i^{(2)}$, which we have introduced, we can solve the problem of estimating the contribution made by each of the random factors to forming the scatter of the function $\phi(V)$.

If $|\delta\Delta\eta_i| \leqslant \varepsilon$, the i-th factor can be regarded as weakly varying, /212 otherwise -- strongly varying. Here ϵ is a prespecified positive number determined by the required precision of computing the dispersion of the function.

Table 6.2

Notation	General formulas	Formulas under normal distr.	Formulas under uniform distribution
$\eta_i^{(1)}$	$a_i^2 M\left[v_i^2\right]$	$a_i^2\sigma_i^2$	$\frac{1}{3} a_i^2 b_i^2$
$\eta_i^{(2)}$	$a_i^2 M\left[v_i^2\right] + a_{ii}^2\left[M\left[v_i^4\right] - \left(M\left[v_i^2\right]\right)^2\right] + M\left[v_i^2\right]\sum_{j=1}^{m}\frac{\overline{a}_{ij}^2}{4} M\left[v_j^2\right]$	$a_i^2\sigma_i^2 + 2a_{ii}^2\sigma_i^4 + \frac{\sigma_i^2}{4}\sum_{j=1}^{m}\overline{a}_{ij}^2\sigma_j^2$	$\frac{1}{3} a_i^2 b_i^2 + \frac{4}{45} a_{ii}^2 b_i^4 + \frac{1}{36} b_i^2 \sum_{j=1}^{m} \overline{a}_{ij}^2 b_j^2$
$\Delta\eta_i^{(2)}$	$a_{ii}^2\left[M\left[v_i^4\right] - \left(M\left[v_i^2\right]\right)^2\right] + M\left[v_i^2\right]\sum_{j=1}^{m}\frac{\overline{a}_{ij}^2}{4} M\left[v_j^2\right]$	$2a_{ii}^2\sigma_i^4 + \frac{\sigma_i^2}{4}\sum_{j=1}^{m}\overline{a}_{ij}^2\sigma_j^2$	$\frac{4}{45} a_{ii}^2 b_i^4 + \frac{1}{36} b_i^2 \sum_{j=1}^{m} a_{ij}^2 b_j^2$

The presence only of the linear model of the function $\phi(V)$ also enables us to solve the problem of differentiating essential and unessential random factors. Thus, we will have the coefficient of significance

$$\delta\eta_i^{(1)} = \frac{\eta_i^{(1)}}{\sigma^2\left[\hat{\varphi}\right]}. \tag{6.12}$$

If $\delta\eta_i^{(1)}$ or $\delta\eta_i^{(2)}$ is smaller than a prespecified number δ, then we can neglect the i-th factor. We must make several remarks concerning the selection of the numbers ϵ and δ.

Usually the realizations of the function $\phi(V)$ on a digital computer are computed with some error $\Delta\phi$. If we assume the error of computation to be random with a normal law of the distribution of probability density and with specified statistical characteristics $M[\Delta\varphi]=0,\ D[\Delta\varphi]=\sigma^2[\Delta\varphi]$, the lower bound of the admissible values of the coefficients of significance can be determined by the following dependents:

$$\varepsilon = \delta = \frac{\sigma^2\left[\Delta\varphi\right]}{\sigma^2\left[\hat{\varphi}\right]}.$$

The upper bound of the admissible values of these coefficients can be specified in accordance with the requirements of the precision with which the statistical characteristics of the function ϕ are computed.

Note that the error in computing the realizations of the function $\phi(V)$ on a digital computer is determined by the error of rounding-off associated with the finite capacity of the arithmetic units of the digital computer, the errors of the method of integration, and the errors associated with the incorrect selection of the integration step.

To use Eq. (6.5) - (6.12), we must set up polynomial (6.3) employing regression, component, dispersion, or factor analysis. Since a fairly large number of studies in the domestic and foreign literature have dealt with a treatment of the latter, it is expedient to set forth the method of stochastic approximation for constructing the polynomial (6.1). The method of stochastic approximation /36/ is associated with stochastic experiment planning, which in large measure meets the specifics of the problems of investigating nonlinear stochastic processes, namely:

randomness of the factors determining the course of the processes (5.2);

a large number of factors, where the known determinate schemes of experiment planning are sufficiently cumbersome in the computational sense; and

use of digital computers for solving problems of investigation.

/213

6.2. Method of Stochastic Approximation

Let us assume that the function $\phi(V)$ under study is a function of the m-dimensional vector of random factors V, whose elements obey a specified symmetric law of distribution of $f_0(V)$ and satisfy the following conditions:

mathematical expectations of the random factors are equal to zero

$$M\left[v_i^{2k+1}\right] = 0 \ (i = 1, 2, \ldots, m), \quad (k = 0, 1, \ldots);$$

the factors are uncorrelated random variables,

$$M\left[v_i v_j\right] = 0 \ (i \neq) = 1, 2, \ldots, m).$$

These assumptions do not diminish the generality of the problem, here enabling us to obtain simpler working relations. Suppose the set Ω_V includes all possible states of the elements of vector V. Then the problem of constructing the approximating polynomial (6.1) for the function $\phi(V)$ can be formulated as a problem in the optimal approximation of the function on the set Ω_V.

The error in approximating the function $\phi(V)$ with the polynomial $\hat{\phi}(V)$ is

$$\varepsilon(V)=\varphi(V)-\hat{\varphi}(V). \tag{6.13}$$

By virtue of the randomness of functions V, we can view the error of approximation as random, and we can consider the statistical characteristics of the error as a measure of the error:

the mathematical expectation of the error

$$J_1=M[\varepsilon]=\int_{\Omega_V}\varepsilon(V)f_0(V)\,dV; \tag{6.14}$$

the second initial moment of the error

$$J_2=M[\varepsilon^2]=\int_{\Omega_V}\varepsilon^2(V)f_0(V)\,dV; \tag{6.15}$$

the second central moment

$$J_3=M[(\varepsilon-M[\varepsilon])^2]=J_2-J_1^2; \tag{6.16}$$

the probability that the error $\epsilon(V)$ does not exceed specified limits $|\varepsilon(V)|\leqslant\varepsilon^0$

$$J_4=P[|\varepsilon(V)|\leqslant\varepsilon^0], \tag{6.17}$$

where ϵ^0 is the specified error of approximation.

We can present further a fairly large number of criteria for the precision of the approximation of function $\phi(V)$ with the polynomial $\hat{\phi}(V)$, however, Eqs. (6.14) - (6.17) are the most applicable from the physical point of view and characterize the value of the mathematical anticipated deviation of the error from the zero value.

Choice of the criterion and its mathematical description in analytic investigations is always subject to criticism. In practical problems the situation is handled much simpler, since the physical meaning of the problem sometimes permits describing the requirements on the problem being solved clearly in a mathemathical sense. Three approaches to forming the criterion are well known in the problem of approximating functions: /214

1) exact coincidence in all experiments (approximation nodes);

2) smallest value of the sum of the squares of the deviations of the polynomial from the approximated function in all experiments (method of least squares); and

3) minimum value of maximum deviation of polynomial $\hat{\phi}(V)$ on the function on the set Ω_V (Chebyshev criterion).

Criteria (6.15) and (6.16) reflect most strongly the requirements on the above-formulated problem of the optimal approximation of the function $\phi(V)$ with a polynomial and can be related to criteria of the precision of approximating the function $\phi(V)$ with the polynomial with the weight of the probability density of the random factors.

Let us find the necessary conditions for a minimum of the criterion for the quality of the precision of approximation (6.15).

Obviously, from the condition

$$\frac{\partial J_2}{\partial a_{ij\ldots\nu}} = 0 \quad (i \leqslant j \leqslant \ldots \leqslant \nu = 1, 2, \ldots, m)$$

we get the following system of algebraic equations:

$$\left.\begin{aligned}
&\frac{1}{2}\frac{\partial J_2}{\partial a_0} = M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_0}\right] = 0;\\
&\frac{1}{2}\frac{J\partial_2}{\partial a_l} = M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_l}\right] = 0 \quad (l = 1, 2, \ldots, m);\\
&\frac{1}{2}\frac{\partial J_2}{\partial a_{kl}} = M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl}}\right] = 0 \quad (k \leqslant l = 1, 2, \ldots, m);\\
&\frac{1}{2}\frac{\partial J_2}{\partial a_{kl\nu}} = M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl\nu}}\right] = 0 \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&\frac{\partial J_2}{\partial a_{kl\nu p}} = M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl\nu p}}\right] = 0 \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m)
\end{aligned}\right\} \tag{6.18}$$

and so on.

The order of the system of algebraic equations (6.18) is obviously determined both by the number of the random factors V as well as by the degree of the approximating polynomial (6.1). /215
Above we have written out a system of equations for determining the coefficients (parameters) of a fourth-degree approximating polynomial. Let us expand the system of equations (6.18), by using the above assumptions and operations.

Since $\frac{\partial \varepsilon}{\partial a_0} = -1,$

$$\left.\begin{aligned}
\frac{\partial \varepsilon}{\partial a_l} &= -v_l \quad (l = 1, 2, \ldots, m);\\
\frac{\partial \varepsilon}{\partial a_{kl}} &= -v_k v_l \quad (k \leqslant l = 1, 2, \ldots, m);\\
\frac{\partial \varepsilon}{\partial a_{kl\nu}} &= -v_k v_l v_\nu \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots m);\\
\frac{\partial \varepsilon}{\partial a_{kl\nu p}} &= -v_k v_l v_\nu v_p \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m),
\end{aligned}\right\} \tag{6.19}$$

then with reference to Eq. (6.13) we can rewrite the system of algebraic equations (6.18) in the form:

$$\left.\begin{aligned}
&M\left[\varphi(V) - \hat{\varphi}(V)\right] = 0;\\
&M\left[(\varphi(V) - \hat{\varphi}(V))v_l\right] = 0 \quad (l = 1, 2, \ldots, m);\\
&M\left[\left(\varphi(V) - \hat{\varphi}(V)\right)v_k v_l\right] = 0 \quad (k \leqslant l = 1, 2, \ldots, m);\\
&M\left[\left(\varphi(V) - \hat{\varphi}(V)\right)v_k v_l v_\nu\right] = 0 \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&M\left[\left(\varphi(V) - \hat{\varphi}(V)\right)v_k v_l v_\nu v_p\right] = 0 \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m)
\end{aligned}\right\}$$

or

$$\left.\begin{aligned}
&M[\varphi(V)] = M\left[\hat{\varphi}(v)\right];\\
&M[\varphi(V)v_l] = M\left[\hat{\varphi}(V)v_l\right] \quad (l = 1, 2, \ldots m);\\
&M[\varphi(V)v_k v_l] = M\left[\hat{\varphi}(V)v_k v_l\right] \quad (k \leqslant l = 1, 2 \ldots, m);\\
&M[\varphi(V)v_k v_l v_\nu] = M\left[\hat{\varphi}(V)v_k v_l v_\nu\right] \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&M[\varphi(V)v_k v_l v_\nu v_p] = M\left[\hat{\varphi}(V)v_k v_l v_\nu v_p\right] \quad (k \leqslant l \leqslant \nu \leqslant p =\\
&= 1, 2, \ldots, m).
\end{aligned}\right\} \tag{6.20}$$

We introduce the notation /216

$$\left.\begin{aligned}
Z_0 &= M[\varphi(V)];\\
Z_l^{(1)} &= M[\varphi(V)v_l] \quad (l = 1, 2, \ldots, m);\\
Z_{kl}^{(2)} &= M[\varphi(V)v_k v_l] \quad (k \leqslant l = 1, 2, \ldots, m);\\
Z_{kl\nu}^{(3)} &= M[\varphi(V)v_k v_l v_\nu] \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
Z_{kl\nu p}^{(4)} &= M[\varphi(V)v_k v_l v_\nu v_p] \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m),
\end{aligned}\right\} \tag{6.21}$$

then we can write the system of equations (6.20) in the form:

$$\left.\begin{aligned}
&Z_0 = M\left[\hat{\varphi}(V)\right];\\
&Z_l^{(1)} = M\left[\hat{\varphi}(V)\,v_l\right] \quad (l = 1, 2, \ldots, m);\\
&Z_{kl}^{(2)} = M\left[\hat{\varphi}(V)\,v_k v_l\right] \quad (k \leqslant l = 1, 2, \ldots, m);\\
&Z_{kl\nu}^{(3)} = M\left[\hat{\varphi}(V)\,v_k v_l v_\nu\right] \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&Z_{kl\nu p}^{(4)} = M\left[\hat{\varphi}(V)\,v_k v_l v_\nu v_p\right] \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m).
\end{aligned}\right\} \tag{6.22}$$

For the criterion (6.16) represented in the form $J_3 = J_2^2 - J_1^2$, we can also obtain a system of equations as follows, analogous to Eqs. (6.22):

$$\left.\begin{aligned}
&\frac{1}{2}\frac{\partial J_3}{\partial a_0} = \frac{1}{2}\frac{\partial J_2}{\partial a_0} - J_1\frac{\partial J_1}{\partial a_0} = 0;\\
&\frac{1}{2}\frac{\partial J_3}{\partial a_l} = \frac{1}{2}\frac{\partial J_2}{\partial a_l} - J_1\frac{\partial J_1}{\partial a_l} = 0 \quad (l = 1, 2, \ldots, m);\\
&\frac{1}{2}\frac{\partial J_3}{\partial a_{kl}} = \frac{1}{2}\frac{\partial J_2}{\partial a_{kl}} - J_1\frac{\partial J_1}{\partial a_{kl}} = 0 \quad (k \leqslant l = 1, 2, \ldots, m);\\
&\frac{1}{2}\frac{\partial J_3}{\partial a_{kl\nu}} = \frac{1}{2}\frac{\partial J_2}{\partial a_{kl\nu}} - J_1\frac{\partial J_1}{\partial a_{kl\nu}} = 0 \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&\frac{1}{2}\frac{\partial J_3}{\partial a_{kl\nu p}} = \frac{1}{2}\frac{\partial J_2}{\partial a_{kl\nu p}} - J_1\frac{\partial J_1}{\partial a_{kl\nu p}} = 0 \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m).
\end{aligned}\right\} \tag{6.23}$$

With reference to Eqs. (6.14) and (6.18), we can transform the system of equations (6.23) and write it as:

$$\left.\begin{aligned}
&M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_0}\right] = M[\varepsilon(V)]\,M\left[\frac{\partial\varepsilon}{\partial a_0}\right];\\
&M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_l}\right] = M[\varepsilon(V)]\,M\left[\frac{\partial\varepsilon}{\partial a_l}\right] \quad (l = 1, 2, \ldots, m);\\
&M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl}}\right] = M[\varepsilon(V)]\,M\left[\frac{\partial\varepsilon}{\partial a_{kl}}\right] \quad (k \leqslant l = 1, 2, \ldots, m)\\
&M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl\nu}}\right] = M[\varepsilon(V)]\,M\left[\frac{\partial\varepsilon}{\partial a_{kl\nu}}\right] \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&M\left[\varepsilon(V)\frac{\partial\varepsilon}{\partial a_{kl\nu p}}\right] = M[\varepsilon(V)]\,M\left[\frac{\partial\varepsilon}{\partial a_{kl\nu p}}\right] \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m).
\end{aligned}\right\} \tag{6.24}$$

/217

Referring to Eqs. (6.14) and (6.19), we can compute:

$$
\left.\begin{aligned}
&M\left[\frac{\partial\varepsilon}{\partial a_0}\right]=-1;\\
&M\left[\frac{\partial\varepsilon}{\partial a_l}\right]=-M[v_l]\quad(l=1,2,\ldots,m);\\
&M\left[\frac{\partial\varepsilon}{\partial a_{kl}}\right]=-M[v_kv_l]\quad(k\leqslant l=1,2,\ldots,m);\\
&M\left[\frac{\partial\varepsilon}{\partial a_{kl\nu}}\right]=-M[v_kv_lv_\nu]\quad(k\leqslant l\leqslant\nu=1,2,\ldots,m);\\
&M\left[\frac{\partial\varepsilon}{\partial a_{kl\nu p}}M\right]=-M[v_kv_lv_\nu v_p]\quad(k\leqslant l\leqslant\nu\leqslant p=1,2,\ldots,m).
\end{aligned}\right\}\tag{6.25}
$$

Inserting Eq. (6.25) into Eqs. (6.24), we get

$$
\left.\begin{aligned}
&M[\varepsilon(V)]=M[\varepsilon(V)];\\
&-Z_l^{(1)}+Z_0M[v_l]=-M\left[\hat{\varphi}(V)v_l\right]+\\
&+M\left[\hat{\varphi}(V)\right]M[v_l]\quad(l=1,2,\ldots,m);\\
&-Z_{kl}^{(2)}+Z_0M[v_kv_l]=-M\left[\hat{\varphi}(V)v_lv_k\right]+M\left[\hat{\varphi}(V)\right]\times\\
&\times M[v_kv_l]\quad(k\leqslant l=1,2,\ldots,m);\\
&-Z_{kl\nu}^{(3)}+Z_0M[v_kv_lv_\nu]=-M\left[\hat{\varphi}(V)v_kv_lv_\nu\right]+M\left[\hat{\varphi}(V)\right]\times\\
&\times M[v_kv_lv_\nu]\quad(k\leqslant l\leqslant\nu=1,2,\ldots,m);\\
&-Z_{kl\nu p}^{(4)}+Z_0M[v_kv_lv_\nu v_p]=-M\left[\hat{\varphi}(V)v_kv_lv_\nu v_p\right]+\\
&+M\left[\hat{\varphi}(V)\right]M[v_kv_lv_\nu v_p]\quad(k\leqslant l\leqslant\nu\leqslant p=1,2,\ldots,m).
\end{aligned}\right\}\tag{6.26}
$$

We can easily show that the first equation of system (6.26) degenerated at identity. This means that the coefficient a_0 cannot be determined from the necessary conditions that the criterion of the precision of approximation (6.16) is optimal. It is defined from the condition of the zero-identity of the mathematical expectation of the approximation error $M[\varepsilon]\equiv 0.$

After referring to the earlier-made assumptions on the statistical characteristics of vector V

$$M\left[v_i^{2k+1}\right]=0\quad(i=1,2,\ldots,m;\ k=0,1,\ldots)$$

the system of equation (6.26) can be written as: /218

$$
\left.\begin{aligned}
&Z_l^{(1)}=M\left[\hat{\varphi}(V)v_l\right]\quad(l=1,2,\ldots,m);\\
&Z_{kl}^{(2)}=M\left[\hat{\varphi}(V)v_kv_l\right]\quad(k\neq l=1,2,\ldots,m);
\end{aligned}\right\}\tag{6.27}
$$

$$
\begin{aligned}
&Z^{(2)}_{kk} - Z_0 M\left[v_k^2\right] = M\left[\hat{\varphi}(V)\, v_k^2\right] - M\left[\hat{\varphi}(V)\right] M\left[v_k^2\right] \quad (k = 1, 2, \ldots, m);\\
&Z^{(3)}_{kl\nu} = M\left[\hat{\varphi}(V)\, v_k v_l v_\nu\right] \quad (k \leqslant l \leqslant \nu = 1, 2, \ldots, m);\\
&Z^{(4)}_{kkkk} - Z_0 M\left[v_k^4\right] = M\left[\hat{\varphi}(V)\, v_k^4\right] - M\left[\hat{\varphi}(V)\right] M\left[v_k^4\right] \quad (k = \\
&= 1, 2, \ldots, m);\\
&Z^{(4)}_{kkkl} = M\left[\hat{\varphi}(V)\, v_k^3 v_l\right] \quad (k < l = 1, 2, \ldots, m);\\
&Z^{(4)}_{kkll} - Z_0 M\left[v_k^2\right] M\left[v_l^2\right] = M\left[\hat{\varphi}(V)\, v_k^2 v_l^2\right] - \\
&- M\left[\hat{\varphi}\right] M\left[v_k^2\right] M\left[v_l^2\right] \quad k < l = 1, 2, \ldots m);\\
&Z^{(4)}_{kll\nu} = M\left[\hat{\varphi}(V)\, v_k v_l^2 v_\nu\right] \quad (k < l < \nu = 1, 2, \ldots, m);\\
&Z^{(4)}_{klll} = M\left[\hat{\varphi}(V)\, v_k v_l^3\right] \quad (k < l = 1, 2, \ldots, m);\\
&Z^{(4)}_{kl\nu p} = M\left[\hat{\varphi}(V)\, v_k v_l v_\nu v_p\right] \quad (k \leqslant l \leqslant \nu \leqslant p = 1, 2, \ldots, m).
\end{aligned}
\qquad \begin{matrix}(6.27)\\ (\text{cont})\end{matrix}
$$

Similar expressions can be written also for the criterion (6.17). Let us consider the right-hand side of the first equation of system of equations (6.22). After necessary transformations, we will have

$$
\begin{aligned}
M\left[\hat{\varphi}(V)\right] &= a_0 + \sum_{i=1}^{m} a_i M\left[v_i\right] + \sum_{\substack{i,\, j=1\\ (i<j)}}^{m} a_{ij} M\left[v_i v_j\right] + \\
&+ \sum_{\substack{i,\, j,\, \nu=1\\ (i<j\leqslant\nu)}}^{m} a_{ij\nu} M\left[v_i v_j v_\nu\right] + \sum_{\substack{i,\, j,\, \nu,\, p=1\\ (i<j\leqslant\nu\leqslant p)}}^{m} a_{ij\nu p} M\left[v_i v_j v_\nu v_p\right].
\end{aligned}
$$

Analogous relations can be written for the right-hand sides of the remaining equations of system (6.24). From the last expression there follows an obvious conclusion. The right-hand sides of system of equations (6.22) are linear functions of the unknown coefficients of polynomial (6.1) and the statistical characteristics of the factors.

We introduce the vectors

$$
\begin{aligned}
Z &= \{Z_0;\ Z_l^{(1)} \quad (l = 1, 2, \ldots, m);\ Z^{(2)}_{kl} \quad (k \leqslant l = 1, 2, \ldots, m); \ldots\},\\
A &= \{a_0;\ a_l \quad (l = 1, 2, \ldots, m);\ a_{kl} \quad (k \leqslant l = 1, 2, \ldots, m); \ldots\}
\end{aligned}
$$

and the block matrix C of the statistical characteristics of the element of vector V, of the form /219

$$C = \left[\begin{array}{c:c:c:c:c} c_{11} & c_{12} & c_{13} & c_{14} & \ldots \\ \hdashline c_{21} & c_{22} & c_{23} & c_{24} & \ldots \\ \hdashline c_{31} & c_{32} & c_{33} & c_{34} & \ldots \\ \hdashline c_{41} & c_{42} & c_{43} & c_{44} & \ldots \\ \hdashline \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right],$$

where $c_{11} = 1$; $c_{12} = c_{21}^* = (M[v_i])$; $c_{13} = c_{31}^* = (M[v_i v_j])$; $c_{14} = c_{41}^* = (M[v_i v_j v_l])$; $c_{22} = (M[v_i v_j])$ and so on.

Then system of equations (6.11) can be written in matrical form

$$Z = CA, \tag{6.28}$$

from whence we can find the vector

$$A = C^{-1}Z,$$

where C^{-1} is the matrix that is the reciprocal of matrix C.

We note one feature of matrix C. By virtue of the above-made assumptions on the statistical characteristics of the factors, the matrix includes blocks of zero matrices $(M[v_i])$, $(M[v_i v_j v_l])$, $(M[v_i v_j v_k v_l v_\nu])$, and so on, therefore it can be represented in the form

$$C = \left[\begin{array}{c:c:c:c:c} c_{11} & 0 & c_{13} & 0 & \ldots \\ \hdashline 0 & c_{22} & 0 & c_{24} & \ldots \\ \hdashline c_{13}^* & 0 & c_{33} & 0 & \ldots \\ \hdashline 0 & c_{24}^* & 0 & c_{44} & \ldots \\ \hdashline \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right].$$

Denoting the matrix C^{-1} by D, and its blocks by d_{ij}, we can represent it in the form

$$D = \left[\begin{array}{c:c:c:c:c} d_{11} & 0 & d_{13} & 0 & \ldots \\ \hdashline 0 & d_{22} & 0 & d_{24} & \ldots \\ \hdashline d_{13}^* & 0 & d_{33} & 0 & \ldots \\ \hdashline 0 & d_{24}^* & 0 & d_{44} & \ldots \\ \hdashline \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right],$$

which repeats the form of matrix C.

Computing the matrix D in general form for any degree of the polynomial does not appear possible; however, from analyzing its form we can make conclusions that are quite important computationally speaking. /220

First of all, when using the Frobenius algorithms /12/ for computing the matrix $D = C^{-1}$, we can organize the procedure of the successive inversion of the matrix consisting of four blocks.

Thus, for polynomials of the first degree we will have

$$C_1 = \left(\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & c_{22} \end{array}\right).$$

For polynomials of the second degree, we can use the matrix

$$C_2 = \left(\begin{array}{c:c:c} \multicolumn{2}{c:}{C_1} & \begin{matrix} c_{13} \\ 0 \end{matrix} \\ \hdashline c_{12}^* & 0 & c_{33} \end{array}\right).$$

For polynomials of the third degree we will have

$$C_3 = \left[\begin{array}{c:c:c:c} \multicolumn{3}{c:}{C_2} & \begin{matrix} 0 \\ c_{24} \\ 0 \end{matrix} \\ \hdashline 0 & c_{24}^* & 0 & c_{44} \end{array}\right]$$

and so on.

The importance of this conclusion follows from the fact that when the Frobenius algorithms are used in computing the reciprocal matrix by partitioning it into blocks of smaller order, we must compute a matrix that is the reciprocal of the matrix standing in one of the blocks. Using as such a matrix the one standing in block (1, 1), we can easily circumvent the process of computing the reciprocal matrix, except for the case of a polynomial of first degree. Actually, this is so by virtue of the absence of the property of complete invariance of the numerical values of the coefficient with respect to the degree of the approximating polynomial.

Secondly, we note that the property of partial invariance of some of the coefficients of the approximating polynomials whose degrees (of the polynomials) differ from unity (Table 6.3).

In Table 6.3, the identical number "asterisk" denotes the coefficients of the approximating polynomials that do not change in value when the degree of the polynomial is raised by one. The property of partial invariance of the coefficients of the approximating polynomials can be successfully used in computing the analytic expressions associating the coefficient of the approximating polynomials, the elements of the vector V, and the statistical characteristics of the vectors.

Table 6.3 /221

Coefficient	Degree of polynomial					
	1	2	3	4	5	6
a_0	*	**	**	****	****	******
a_i	*	*	***	***	*****	*****
a_{ij}	—	**	**	****	****	******
a_{ijk}	—	—	***	***	*****	*****
a_{ijkl}	—	—	—	****	****	******
a_{ijklv}	—	—	—	—	*****	*****
a_{ijklvp}	—	—	—	—	—	******

Let us dwell further on deriving working formulas of this relationship. For the moment we will assume that the elements of vector Z are known. We will dwell below on the physical meaning of the elements of vector Z and the algorithms of their computation.

6.3. Constructing Polynomials of First, Second, and Third Degrees in Stochastic Experiment Planning

The aim of this section is to find an explicit relationship between the elements of vector A (see (6.28)), the elements of vector Z, and the statistical characteristics of the factors. When solving this problem for a first-degree polynomial

$$\hat{\varphi} = a_0 + \sum_{i=1}^{m} a_i v_i \qquad (6.29)$$

we must use the first two equations of system (6.22). We will have

$$\begin{aligned} Z_0 &= M\left[\hat{\varphi}(V)\right], \\ Z_l^{(1)} &= M\left[\hat{\varphi}(V)\,v_l\right] \quad (l = 1, 2, \ldots, m). \end{aligned} \tag{6.30}$$

After polynomial (6.29) is substituted into expressions (6.30) and the transformations are carried out, we will have the following system of algebraic equations:

$$\begin{aligned} Z_0 &= a_0 + \sum_{i=1}^{m} a_i M[v_i], \\ Z_l^{(1)} &= a_0 M[v_l] + \sum_{i=1}^{m} a_i M[v_i v_l] \quad (l = 1, 2, \ldots, m) \end{aligned}$$

or

$$\begin{aligned} Z_0 &= a_0, \\ Z_l^{(1)} &= a_l M[v_l^2] \quad (l = 1, 2, \ldots, m), \end{aligned} \tag{6.31}$$

from whence we get expressions for computing the coefficients: /222

$$a_0 = Z_0, \tag{6.32}$$

$$a_l = \frac{Z_{(l)}^{(1)}}{M\left[v_l^2\right]} \quad (l = 1, 2, \ldots, m). \tag{6.33}$$

Thus, the problem of constructing first-degree polynomials has been solved. The coefficients a_o, a_l $(l = 1, 2, \ldots, m)$ are expressed in terms of the elements of vector Z by the fairly simple Eqs. (6.32) and (6.33).

To construct the second-degree polynomial

$$\hat{\varphi} = a_0 + \sum_{i=1}^{m} a_i v_i + \sum_{\substack{i, j=1 \\ (i<j)}}^{m} a_{ij} v_i v_j \tag{6.34}$$

we must use the first three equations of system (6.22):

$$\begin{aligned} Z_0 &= M\left[\hat{\varphi}(V)\right]; \\ Z_l^{(1)} &= M\left[\hat{\varphi}(V)\,v_l\right] \quad (l = 1, 2, \ldots, m); \\ Z_{kl}^{(2)} &= M\left[\hat{\varphi}(V)\,v_k v_l\right] \quad (k \leqslant l = 1, 2, \ldots, m). \end{aligned} \tag{6.35}$$

Substituting polynomial (6.34) into Eqs. (6.35) and carrying out all the necessary transformations, we can write

$$\left.\begin{aligned}
Z_0 &= a_0 + \sum_{l=1}^{m} a_l M[v_l] + \sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij} M[v_i v_j];\\
Z_l^{(1)} &= a_0 M[v_l] + \sum_{l=1}^{m} a_l M[v_l v_l] + \sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij} M[v_l v_i v_j] \quad (l = 1, 2, \ldots, m),\\
Z_{kl}^{(2)} &= a_0 M[v_k v_l] + \sum_{l=1}^{m} a_l M[v_k v_l v_l] + \sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij} M[v_k v_l v_i v_j]
\end{aligned}\right\}$$

or

$$\left.\begin{aligned}
Z_0 &= a_0 + \sum_{l=1}^{m} a_{ll} M[v_l^2];\\
Z_l^{(1)} &= a_l M[v_l^2] \quad (l = 1, 2, \ldots, m);\\
Z_{kl}^{(2)} &= a_0 M[v_k v_l] + \sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij} M[v_k v_l v_i v_j] \quad (k \leqslant l = 1, 2, \ldots, m).
\end{aligned}\right\} \tag{6.36}$$

From system of equations (6.36) we single out the second equation (which deals with the invariance of the coefficients a_l (l = 1, 2, ..., m) for the second degree of the polynomial). The first and second equations can be represented in matrical form (6.28), by introducing the notation

$$C = \left(\begin{array}{c|c} 1 & c_{11} \\ \hline c_{12}^{*} & c_{12} \end{array}\right),$$ /223

where

$$c_{12} = \Big(\overbrace{M[v_1^2]\,0\ldots0}^{m}\ \overbrace{M[v_2^2]\,0\ldots0}^{m-1}\ \ldots\ \overbrace{M[v_m^2]}^{1}\Big),$$

$$C_{22} = \begin{array}{cc}
 & \begin{array}{ccc} \overbrace{\hphantom{M[v_1^2]M[v_2^2]\ 0\ \ldots\ M[v_1^2]M[v_m^2]}}^{m} & \overbrace{\hphantom{M[v_1^2]M[v_2^2]\ 0\ \ldots\ 0}}^{m-1} \ \ldots & \overbrace{\hphantom{M[v_1^2]M[v_m^2]}}^{1} \end{array} \\
\begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ m-1\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \end{array} &
\left[\begin{array}{cccc:cccc:c:c}
M[v_1^4] & 0 & \ldots & 0 & M[v_1^2]M[v_2^2] & 0 & \ldots & 0 & \ldots & M[v_1^2]M[v_m^2] \\
0 & M[v_1^2]M[v_2^2] & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & M[v_1^2]M[v_m^2] & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\hdashline
M[v_1^2]M[v_2^2] & 0 & \ldots & 0 & M[v_2^4] & 0 & \ldots & 0 & \ldots & M[v_2^2]M[v_m^2] \\
0 & 0 & \ldots & 0 & 0 & M[v_2^2]M[v_3^2] & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\hdashline
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\hdashline
M[v_1^2]M[v_m^2] & 0 & \ldots & 0 & M[v_2^2]M[v_m^2] & 0 & \ldots & 0 & \ldots & M[v_m^4]
\end{array}\right].
\end{array}$$

To invert matrix C, we use the Frobenius algorithm for the inversion of a block matrix: /224

$$D=\left(\begin{array}{c|c} c_{11}^{-1}+c_{11}^{-1}c_{12}H^{-1}c_{12}^{*}c_{11}^{-1} & -c_{11}^{-1}c_{12}H^{-1} \\ \hline -H^{-1}c_{12}^{*}c_{11}^{-1} & H^{-1} \end{array}\right),$$

where

$$H=c_{22}-c_{12}^{*}c_{11}^{-1}c_{12}.$$

Since $c_{11}^{-1}=1$, we get the following working formulas:

$$H=c_{22}-c_{12}^{*}c_{12};$$

$$D=\left(\begin{array}{c|c} 1+c_{12}H^{-1}c_{12}^{*} & -(H^{-1}c_{12}^{*})^{*} \\ \hline -H^{-1}c_{12}^{*} & H^{-1} \end{array}\right). \qquad (6.37)$$

We present the necessary transformations.

Since

$$c_{12}^{*}c_{12}=$$

$$=\begin{array}{cc}
 & \begin{array}{cccc} \overbrace{\hphantom{(M[v_1^2])^2\ 0\ \ldots\ 0}}^{m} & \overbrace{\hphantom{M[v_1^2]M[v_2^2]\ 0\ \ldots\ 0}}^{m-1} & \ldots & \overbrace{\hphantom{M[v_1^2]M[v_m^2]}}^{1} \end{array} \\
\begin{array}{c} m \\ \\ m-1 \\ \\ \vdots \\ 1 \end{array} &
\left[\begin{array}{cccc|cccc|c|c}
(M[v_1^2])^2 & 0 & \ldots & 0 & M[v_1^2]\,M[v_2^2] & 0 & \ldots & 0 & \ldots & M[v_1^2]\,M[v_m^2] \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\hline
M[v_1^2]\,M[v_2^2] & 0 & \ldots & 0 & (M[v_2^2])^2 & 0 & \ldots & 0 & \ldots & M[v_2^2]\,M[v_m^2] \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\hline
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\hline
M[v_1^2]\,M[v_m^2] & 0 & \ldots & 0 & M[v_2^2]\,M[v_m^2] & 0 & \ldots & 0 & \ldots & (M[v_m^2])^2
\end{array}\right]
\end{array}$$

then the matrix H is a diagonal matrix of the form

$$H=\operatorname{diag}\{M[v_1^4]-(M[v_1^2])^2,\ M[v_1^2]\,M[v_2^2],\ \ldots,\ M[v_1^2]\,M[v_m^2],\ M[v_2^4]-(M[v_2^2])^2,\ M[v_2^2]\,M[v_3^2],\ \ldots,\ M[v_m^4]-(M[v_m^2])^2\}.$$

The matrix that is the reciprocal of H obviously is of the form

$$H^{-1} = \operatorname{diag}\left\{\frac{1}{M\left[v_1^4\right]-\left(M\left[v_1^2\right]\right)^2},\ \frac{1}{M\left[v_1^2\right]M\left[v_2^2\right]},\ \ldots,\ \frac{1}{M\left[v_2^4\right]-\left(M\left[v_2^2\right]\right)^2},\ \frac{1}{M\left[v_2^2\right]M\left[v_3^2\right]},\ \ldots,\ \frac{1}{M\left[v_m^4\right]-\left(M\left[v_m^2\right]\right)^2}\right\}. \tag{6.38}$$

Using Eqs. (6.37) and (6.38), we can easily get /225

$$d_{12}^{*} = \left(\frac{-M\left[v_1^2\right]}{M\left[v_1^4\right]-\left(M\left[v_1^2\right]\right)^2}\,0\ \ldots\ \ldots\,0\,\frac{-M\left[v_2^2\right]}{M\left[v_2^4\right]-\left(M\left[v_2^2\right]\right)^2}\,0\ \ldots\ \ldots\,0\ \ldots\,\frac{-M\left[v_m^2\right]}{M\left[v_m^4\right]-\left(M\left[v_m^2\right]\right)^2}\right). \tag{6.39}$$

Using Eqs. (6.39) and matrix (6.37), let us write out the working formulas for computing the coefficients of regression a_0 and a_{ij} $(i\leqslant j=1, 2, \ldots, m)$:

$$a_0 = \left[1+\sum_{i=1}^{m}\frac{\left(M\left[v_i^2\right]\right)^2}{\Delta_1^i}\right]Z_0-\sum_{i=1}^{m}\frac{M\left[v_i^2\right]Z_{ii}^{(2)}}{\Delta_1^i}, \tag{6.40}$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ij}^2}{M\left[v_i^2\right]M\left[v_j^2\right]}, & i\leqslant j=1,2,\ldots,m; \quad (6.41)\\[2ex] \dfrac{Z_{ii}^{(2)}-M\left[v_i^2\right]Z_0}{\Delta_1^i}, & i=1,2,\ldots,m; \quad (6.42)\end{cases}$$

where

$$\Delta_1^i = M\left[v_i^4\right]-\left(M\left[v_i^2\right]\right)^2 \quad (i=1,2,\ldots,m). \tag{6.43}$$

Based on these Eqs. (6.40), (6.33), (6.41), and (6.42), we can write out the working formulas for computing the coefficients of the second-degree approximating polynomial:

$$a_0 = \left[1+\sum_{i=1}^{m}\frac{\left(M\left[v_i^2\right]\right)^2}{\Delta_1^i}\right]Z_0-\sum_{i=1}^{m}\frac{M\left[v_i^2\right]}{\Delta_1^i}Z_{ii}^{(2)},$$

$$a_i = \frac{Z_i^{(1)}}{M\left[v_i^2\right]} \quad (i=1,2,\ldots,m), \tag{6.44}$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ji}^{(2)}}{M\left[v_i^2\right] M\left[v_j^2\right]}, & i < j = 1, 2, \ldots, m; \\ \dfrac{Z_{ii}^{(2)} - M\left[v_i^2\right] Z_0}{\Delta_1^i}, & i = 1, 2, \ldots, m. \end{cases} \tag{6.44 (cont)}$$

Of interest is the case when the statistical characteristics /226 of the factors are equal to each other. With $M\left[v_i^{2k}\right] = M\left[v^{2k}\right] (i = 1, 2, \ldots, m)$ Eqs. (6.44) become

$$\begin{gathered} a_0 = \left[1 + \frac{m\left(M\left[v^2\right]\right)^2}{\Delta_1}\right] Z_0 - \frac{M\left[v^2\right]}{\Delta_1} \sum_{i=1}^{m} Z_{ii}^{(2)}, \\ a_i = \frac{Z_i^{(1)}}{M\left[v^2\right]}, \ (i = 1, 2, \ldots, m); \\ a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{\left(M\left[v^2\right]\right)^2}, & i < j = 1, 2, \ldots, m; \\ \dfrac{Z_{ii}^{(2)} - M\left[v^2\right] Z_0}{\Delta_1}, & i = 1, 2, \ldots, m. \end{cases} \end{gathered} \tag{6.45}$$

Here Eq. (6.43) is of the form

$$\Delta_1 = M\left[v^4\right] - \left(M\left[v^2\right]\right)^2.$$

To construct the third-degree polynomial

$$\hat{\varphi} = a_0 + \sum_{i=1}^{m} a_i v_i + \sum_{\substack{i, j=1 \\ (i<j)}}^{m} a_{ij} v_i v_j + \sum_{\substack{i, j, \delta=1 \\ (i<j<\delta)}}^{m} a_{ij\delta} v_i v_j v_\delta \tag{6.46}$$

we must take four equations of system (6.22):

$$\begin{gathered} Z_0 = M\left[\hat{\varphi}(V)\right], \\ Z_l^{(1)} = M\left[\hat{\varphi}(V) v_l\right] \ (l = 1, 2, \ldots, m), \\ Z_{kl}^{(2)} = M\left[\hat{\varphi}(V) v_k v_l\right] \ (k \leqslant l = 1, 2, \ldots, m), \\ Z_{kl\nu}^{(3)} = M\left[\hat{\varphi}(V) v_k v_l v_\nu\right] \ (k \leqslant l \leqslant \nu = 1, 2, \ldots, m). \end{gathered}$$

Since the first and third equations of this system written above were used in setting up the second-degree polynomial (6.34) and enables us to determine the coefficients a_o and a_{ij} $(i \leqslant j = 1, 2, \ldots, m)$, to get the working formulas for the coefficients a_i $(i = 1, 2, \ldots, m)$, a_{ijk} $(i \leqslant j \leqslant k = 1, 2, \ldots, m)$ we must use the second and fourth equations of this system (6.22).

Let us write out the second and fourth equations of the system /227

$$\left.\begin{aligned} Z_l^{(1)} &= a_l M\left[v_l^2\right] + \sum_{\substack{l, j, k=1 \\ (l \leqslant j \leqslant k)}}^{m} a_{ljk} M\left[v_l v_i v_j v_k\right] \quad (l = 1, 2, \ldots, m); \\ Z_{l\nu\delta}^{(3)} &= \sum_{l=1}^{m} a_l M\left[v_l v_\nu v_\delta v_l\right] + \sum_{\substack{l, j, k=1 \\ (l \leqslant j \leqslant k)}}^{m} a_{ljk} M\left[v_l v_\nu v_\delta v_l \times \right. \\ &\left. \times v_j v_k\right] \quad (l \leqslant \nu \leqslant \delta = 1, 2, \ldots, m). \end{aligned}\right\}$$

We can represent the resulting system of equations for convenience of computation in the expanded form:

$$Z_l^{(1)} = a_l M\left[v_l^2\right] + a_{lll} M\left[v_l^4\right] + \sum_{\substack{j=1 \\ (j \neq l)}}^{m} a_{ljj} M\left[v_j^2\right] M\left[v_l^2\right] \quad (l = 1, 2, \ldots, m); \tag{6.47}$$

$$Z_{lll}^{(3)} = a_l M\left[v_l^4\right] + a_{lll} M\left[v_l^6\right] + \sum_{\substack{j=1 \\ (j \neq l)}}^{m} a_{ljj} M\left[v_j^2\right] M\left[v_l^4\right] \quad (i = 1, 2, \ldots, m); \tag{6.48}$$

$$Z_{iij}^{(3)} = a_j M\left[v_i^2\right] M\left[v_j^2\right] + a_{iij} M\left[v_i^4\right] M\left[v_j^2\right] + a_{jjj} M\left[v_i^2\right] M\left[v_j^4\right] + \sum_{\substack{k=1 \\ (k \neq i \neq j)}}^{m} a_{jkk} M\left[v_i^2\right] M\left[v_j^2\right] M\left[v_k^2\right] \quad (i < j = 1,2, \ldots, m); \tag{6.49}$$

$$Z_{ijl}^{(3)} = a_{ijl} M\left[v_i^2\right] M\left[v_j^2\right] M\left[v_l^2\right] \quad (i < j < l = 1,2, \ldots, m); \tag{6.50}$$

$$Z_{ijj}^{(3)} = a_i M\left[v_i^2\right] M\left[v_j^2\right] + a_{iii} M\left[v_i^4\right] M\left[v_j^2\right] + a_{ijj} M\left[v_i^2\right] M\left[v_j^4\right] + \sum_{\substack{k=1 \\ (k \neq i \neq j)}}^{m} a_{ikk} M\left[v_i^2\right] M\left[v_j^2\right] M\left[v_k^2\right] \quad (i < j = 1,2, \ldots, m). \tag{6.51}$$

From Eq. (6.47) let us determine the expression for the coefficients a_i ($i = 1, 2, \ldots, m$):

$$a_i = \frac{Z_i^{(1)} - a_{iii} M[v_i^4] - \sum_{\substack{j=1 \\ (j \neq i)}}^{m} a_{ijj} M[v_i^2] M[v_j^2]}{M[v_i^2]}. \tag{6.52}$$

and let us substitute it into Eq. (6.48). We get /228

$$Z_{iii}^{(3)} - \frac{M[v_i^4]}{M[v_i^2]} Z_i^{(1)} = a_{iii} \left\{ M[v_i^6] - \frac{(M[v_i^4])^2}{M[v_i^2]} \right\} +$$
$$+ M[v_i^4] \left\{ \sum_{\substack{j=1 \\ (j \neq i)}}^{m} [a_{ijj} M[v_j^2] - a_{ijj} M[v_j^2]] \right\} \quad (i = 1, 2, \ldots, m),$$

whence we will have the expression for computing the coefficients a_{iii}

$$a_{iii} = \frac{Z_{iii}^{(3)} M[v_i^2] - Z_i^{(1)} M[v_i^4]}{\Delta_2^i} \quad (i = 1, 2, \ldots, m), \tag{6.53}$$

where

$$\Delta_2^i = M[v_i^6] M[v_i^2] - (M[v_i^4])^2. \tag{6.54}$$

Substituting Eqs. (6.52) into Eq. (6.49), after uncomplicated transformations we will have the expressions for determining the coefficients a_{iij}:

$$a_{iij} = \frac{Z_{ij}^{(3)} - M[v_i^2] Z_j^{(1)}}{M[v_j^2] \Delta_1^i} \quad (i \neq j = 1, 2, \ldots, m). \tag{6.55}$$

Using algorithms (6.53) and (6.55) and Eq. (6.52), we get the working formula for computing the coefficients

$$a_i = \frac{Z_i^{(1)} \Delta_3^i}{M[v_i^2]} - \frac{Z_{iii}^{(3)} M[v_i^4]}{\Delta_2^i} -$$
$$- \frac{1}{M[v_i^2]} \sum_{\substack{j=1 \\ (j \neq i)}}^{m} \frac{M[v_j^2] Z_{ijj}^{(3)}}{\Delta_1^j} \quad (i = 1, 2, \ldots, m), \tag{6.56}$$

where

$$\Delta_3^i = \frac{M[v_i^6] M[v_i^2]}{\Delta_2^i} - \sum_{\substack{j=1 \\ (j \neq i)}}^{m} \frac{(M[v_j^2])^2}{\Delta_1^j}. \tag{6.57}$$

And, finally, from Eq. (6.50) it follows that

$$a_{ijl} = \frac{Z_{ijl}^{(3)}}{M\left[v_i^2\right] M\left[v_j^2\right] M\left[v_l^2\right]} \quad (i<j<l=1,2,\ldots,m) \tag{6.58}$$

Based on the resulting Eqs. (6.40), (6.56), (6.41), (6.42), (6.43), (6.55), and (6.58) we can write out working formulas for computing the coefficients of the approximating polynomial of /229 third degree:

$$\begin{gathered}
a_0 = \left[1 + \sum_{i=1}^{m} \frac{\left(M\left[v_i^2\right]\right)^2}{\Delta_1^i}\right] Z_0 - \sum_{i=1}^{m} \frac{M\left[v_i^2\right]}{\Delta_1^i} Z_{ii}^{(2)}; \\
a_i = \frac{Z_i^{(1)}\Delta_3^i}{M\left[v_i^2\right]} - \frac{Z_{iii}^{(3)} M\left[v_i^4\right]}{\Delta_2^i} - \frac{1}{M\left[v_i^2\right]} \sum_{\substack{j=1 \\ (j\neq i)}}^{m} \frac{M\left[v_j^2\right]}{\Delta_1^j} Z_{ijj}^{(3)} \quad (i=1,2\ldots,m); \\
a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{M\left[v_i^2\right] M\left[v_j^2\right]}, & i<j=1,2,\ldots,m; \\[2ex] \dfrac{Z_{ii}^{(2)} - M\left[v_i^2\right] Z_0}{\Delta_1^i}, & i=1,2,\ldots,m; \end{cases} \\
a_{ijl} = \begin{cases} \dfrac{Z_{iii}^{(3)} M\left[v_i^2\right] - M\left[v_i^4\right] Z_i^{(1)}}{\Delta_2^i}, & (i=1,2,\ldots,m); \\[2ex] \dfrac{Z_{ij}^{(3)} - M\left[v_i^2\right] Z_j^{(1)}}{M\left[v_j^2\right] \Delta_1^i}, & (i<j=1,2,\ldots,m); \\[2ex] \dfrac{Z_{ijl}^{(3)}}{M\left[v_i^2\right] M\left[v_j^2\right] M\left[v_l^2\right]} & (i<j<l=1,2,\ldots,m). \end{cases}
\end{gathered} \tag{6.59}$$

For equal statistical characteristics of the factors, Eqs. (6.59) become as follows for the coefficients of the third-degree polynomial:

$$\begin{gathered}
a_0 = \left[1 + \frac{M\left[v^2\right]}{\Delta_1} m\right] Z_0 - \frac{M\left[v^2\right]}{\Delta_1} \sum_{i=1}^{m} Z_{ii}^{(2)}; \\
a_i = \frac{\Delta_3 Z_i^{(1)}}{M\left[v^2\right]} - \frac{Z_{iii}^{(3)} M\left[v^4\right]}{\Delta_2} - \frac{1}{\Delta_1} \sum_{\substack{j=1 \\ (j\neq i)}}^{m} Z_{ijj}^{(3)} \quad (i=1,2,\ldots,m); \\
a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{\left(M\left[v^2\right]\right)^2} & (i<j=1,2,\ldots,m); \\[2ex] \dfrac{Z_{ii}^{(2)} - M\left[v^2\right] Z_0}{\Delta_1} & (i=j=1,2,\ldots,m); \end{cases}
\end{gathered} \tag{6.60}$$

$$\left\{\begin{array}{l} \dfrac{Z_{iii}^{(3)} - M[v^2]\,M[v^4]\,Z_i^{(1)}}{\Delta_2} \quad (i = 1,2,\ldots,m); \\ \dfrac{Z_{iij}^{(3)} - M[v^2]\,Z_j^{(1)}}{M[v^2]\,\Delta_1} \quad (i < j = 1,2,\ldots,m); \\ \dfrac{Z_{ijl}^{(3)}}{(M[v^2])^3} \quad (i < j < l = 1,2,\ldots,m). \end{array}\right. \tag{6.60 (cont)}$$

Here Eqs. (6.54) and (6.57) become: /230

$$\Delta_2 = M[v^6]\,M[v^2] - (M[v^4])^2,$$
$$\Delta_3 = \frac{M[v^6]\,M[v^2]}{\Delta_2} + \frac{m-1}{\Delta_1}(M[v^2])^2. \tag{6.61}$$

Thus, for polynomials of the first (6.29), second (6.34), and third (6.44) degrees, we have obtained working formulas for computing the coefficients that relate elements of the vector V and the statistical characteristics of the vectors. Working formula (6.59) are considerably simplified for the case of equal statistical characteristics of factors, as follows from Eqs. (6.60).

Thus, we have obtained working formulas for computing the coefficients of the third-degree approximating polynomials. They are quite simple, especially for identical statistical characteristics of the factors.

The case of polynomials of higher degree is also of interest for practice, however the cumbersomeness of this procedure does not enable us to derive the corresponding working formulas for polynomials of degree higher than the third. Of the stochastic schemes of experiment planning are the set Ω_V, we can single out two schemes that satisfy the assumptions made at the beginning of this chapter:

the scheme with normal distribution of experiments on the set Ω_V; and

the scheme with uniform distribution of experiments on the set Ω_V.

Both schemes can be used in practical calculations, therefore, by using the above-presented formulas for the general case of stochastic experiment planning, we can derive working formulas for computing the coefficients of the approximating polynomials for the schemes of normal and uniform experiment planning on the specified set Ω_V.

Above we obtained working formulas for computing the coefficients of approximating polynomial (6.1) that relate the statistical characteristics of factors to the elements of vector Z. Let us find the error of the stochastic approximation for a specified distribution of the probability density of factors. We can write the expression in the form

$$J_2 = M[\varphi^2(V)] - 2M\left[\varphi(V)\hat{\varphi}(V)\right] + M\left[\hat{\varphi}^2(V)\right]. \tag{6.62}$$

for the criterion (6.15).after transformations.

Let us obtain working formulas for computing the criterion (6.62) in particular cases. For the first-degree polynomial (6.29),Eq. (6.62) becomes /231

$$J_2^{(1)} = M[\varphi^2(V)] - 2a_0 Z_0 - -2\sum_{i=1}^{m} a_i Z_i^{(1)} + a_0^2 + \sum_{i=1}^{m} a_i^2 M[v_i^2]. \tag{6.63}$$

Substituting Eqs. (6.32) and (6.33) for the coefficients a_o and a_i into Eq. (6.100), we will have

$$J_2^{(1)} = M[\varphi^2(V)] - Z_0^2 - \sum_{i=1} \frac{(Z_i^{(1)})^2}{M[v_i^2]}. \tag{6.64}$$

For second-degree polynomial (6.34), Eq. (6.62) can be represented in the form

$$\begin{aligned} J_2^{(2)} = M[\varphi^2(V)] - 2a_0 Z_0 - 2\sum_{i=1}^{m} a_i Z_i^{(1)} - \\ -2\sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij} Z_{ij}^{(2)} + a_0^2 + 2a_0 \sum_{i=1}^{m} a_{ii} M[v_i^2] + \sum_{i=1}^{m} a_i^2 M[v_i^2] + \\ + \sum_{i=1}^{m} a_{ii}^2 M[v_i^4] + \sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ij}^2 M[v_i^2] M[v_j^2] + \\ +2\sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ii} a_{jj} M[v_i^2] M[v_j^2]. \end{aligned} \tag{6.65}$$

Substituting into Eq. (6.65) Eqs. (6.44) for the coefficients a_o, a_i, and a_{ij} $(i \leqslant j = 1, 2, \ldots, m)$ and carrying out the transformations, we get

$$J_2^{(2)} = M[\varphi^2(V)] - \sum_{i=1}^{m} \frac{(Z_i^{(1)})^2}{M[v_i^2]} - \sum_{\substack{i,j=1\\(i<j)}}^{m} \frac{(Z_{ij}^{(2)})^2}{M[v_i^2]M[v_j^2]} - 2a_0Z_0 + a_0^2 + \sum_{i=1}^{m} [2a_0a_{ii}M[v_i^2] + a_{ii}^2M[v_i^4] - 2a_{ii}Z_{ii}^{(2)}] + 2\sum_{\substack{i,j=1\\(i<j)}}^{m} a_{ii}a_{jj}M[v_i^2]M[v_j^2],$$

which can lead to the form /232

$$J_2^{(2)} = M[\varphi^2(V)] - \sum_{i=1}^{m} \frac{(Z_i^{(1)})^2}{M[v_i^2]} - \sum_{\substack{i,j=1\\(i<j)}}^{m} \frac{(Z_{ij}^{(2)})^2}{M[v_i^2]M[v_j^2]} - \sum_{i=1}^{m}\left[\frac{(Z_{ii}^{(2)})^2}{\Delta_1^i} + \frac{(M[v_i^2])^2}{\Delta_1^i}Z_0^2 - Z_0Z_{ii}^{(2)}\frac{M[v_i^2]}{\Delta_1^i}\right] + 2Z_0\sum_{\substack{i,j=1\\(i<j)}}^{m} \frac{M[v_i^2]M[v_j^2]}{\Delta_1^i\Delta_1^j}[M[v_i^2]Z_{jj}^{(2)} - M[v_j^2]Z_{ii}^{(2)}] - Z_0^2. \quad (6.66)$$

For the normalized vector V, Eq. (6.66) is simplified and is of the form

$$J_2^{(2)} = M[\varphi^2(V)] - \frac{1}{M[v^2]}\sum_{i=1}^{m}(Z_i^{(1)})^2 - \frac{1}{(M[v^2])^2}\sum_{\substack{i,j=1\\(i<j)}}^{m}(Z_{ij}^{(2)})^2 - \frac{1}{\Delta_1}\sum_{i=1}^{m}(Z_{ii}^{(2)})^2 - \left(1 + m\frac{(M[v^2])^2}{\Delta_1}\right)Z_0^2 + \frac{2Z_0M[v^2]}{\Delta_1}\sum_{i=1}^{m}Z_{ii}^{(2)} + 2Z_0\frac{(M[v^2])^3}{\Delta_1}\sum_{\substack{i,j=1\\(i<j)}}^{m}(Z_{jj} - Z_{ii}). \quad (6.67)$$

We can similarly obtain expressions for computing the error of the stochastic approximation for higher degrees of the approximating polynomial.

6.5. Computation of Estimates of Moments Z

In the stochastic approximation of the function ϕ (V) with a polynomial of the form (6.1), the statistical characteristics of the function

$$Z_i^{(1)}=M[\varphi(V)v_i],\quad Z_{ij}^{(2)}=M[\varphi(V)v_iv_j],\ldots \tag{6.68}$$

were introduced into consideration.

The exact computation of the moments $Z_0, Z_i^{(1)}, Z_{ij}^{(2)},\ldots$ for a nonlinear model of the process having the form (5.2) is impossible in practice. Therefore it appears useful to employ estimates of the moments Z computed with approximate methods of statistical analysis of nonlinear systems, for example, the method of statistical tests.

If the sequence (5.78) is constructed for a random vector and if for it the sequence of functions

$$\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(N)}, \tag{6.69}$$

is computed, the computation of the estimates of moments (6.68) can be done by employing formulas from the method of statistical tests: /233

$$\begin{gathered}
\left(\hat{Z}_0\right)_N \approx \left(M\left[\hat{\varphi}\right]\right)_N=\frac{1}{N}\sum_{l=1}^{N}\varphi^{(l)};\\
\left(\hat{Z}_i^{(1)}\right)_N \approx \left(M\left[\hat{\varphi}(V)v_i\right]\right)_N=\frac{1}{N}\sum_{j=1}^{N}\varphi^{(j)}\left[V^{(j)}\right]v^{(l)}\quad (i=1,2,\ldots,m);\\
\left(\hat{Z}_{ij}^{(2)}\right)_N \approx \left(M\left[\hat{\varphi}(V)v_iv_j\right]\right)_N=\frac{N-1}{N}\left(\hat{Z}_{ij}^{(2)}\right)_{N-1}+\\
+\frac{1}{N}\varphi^{(N)}v_i^{(N)}v^{(N)}.
\end{gathered} \tag{6.70}$$

Similar formulas can be written for computing the estimates of the introduced moments of higher degree by using the method of statistical tests.

We can easily see that computing the estimates of moment (6.68) and the initial moment $M[\varphi(V)]$ of the function ϕ(V) can make use of the same sequences (5.78) and (6.69).

Here the sequence (5.78) can be constructed in digital computers employing standard programs of random numbers with assigned density of the probability distribution.

Example 6.1. As an illustration of the method of stochastic approximation, let us look at an example of investigating the effect of random perturbations (elements of a canonical expansion of the random components of air density) in a process of the descent made by a flight vehicle in the dense atmospheric layers of the earth described by nonlinear equations from /13/:

$$\begin{aligned} \dot{V} &= -k_1\rho(h)V^2 - k_2\sin\theta \quad (k_1 = 1{,}75\cdot 10^{-3}); \\ \dot{\theta} &= -\frac{k_2}{V}\cos\theta \quad (k_2 = 0{,}028); \\ \dot{x} &= V\cos\theta; \quad \dot{y} = V\sin\theta; \quad h = y - \frac{x^2}{R}; \\ \rho(h) &= \rho_0(h) + \Delta\rho(h)\mu(h). \end{aligned} \tag{6.1.1}$$

The symbols used in setting up Eq. (6.1.1) are given in example (5.1).

Let us consider as the perturbation the normalized random component of the deviation of atmospheric density from the standard $\rho_0(h)$, which we specify in the form of a segment of a Fourier series (see Chapter Four) with random coefficients

$$\Delta\rho(h) = \sum_{i=0}^{6} [\beta_{2i+1}v_{2i+1}\cos(i+1)\omega h + \beta_{2i}v_{2i}\sin(i+1)\omega h],$$
$$\omega = 9\cdot 10^{-4}\left(\frac{1}{\mu}\right),$$

where $\beta_i\ (i=1, 2, \ldots, 14)$ are weighting coefficients given in Table 6.4.

Table 6.4

n	1	2	3	4	5	6	7
β	0,69	0,46	0,31	0,21	0,19	0,17	0,16
n	8	9	10	11	12	13	14
β	0,69	0,46	0,31	0,21	0,19	0,17	0,16

The numerical value of the normalizing cofactor $\mu(h)$ used in the numerical calculations is given in Table 6.5. /234

Table 6.5

h, km	0,5	0	1	2	3	5	7	10
$\mu(h)\cdot 10^3$	4078	3263	2090	1,428	1,019	0,659	0,600	0,610
h km	15	20	25	30	35	40	45	50
$\mu(h)\cdot 10^4$	7915	2010	2703	0,928	0,504	0,509	0,179	0,110

We will assume that the characteristics of the random factors are assigned and are equal to

$$M[v_i] = 0; \quad M[v_i, v_j] = \begin{cases} \sigma_i^2 = 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

The investigation was conducted for the deviation of the coordinate x of the perturbing motion from the ℓℓ value in unperturbed motion ($\Delta\rho = 0$) at the altitude h = 100 m for the following initial conditions of the system of equations (6.1.1): $V_0 = 7850$ m/c; $\theta_0 = -5°$; $y_0 = 50$ km; and $x_0 = 0$.

The statistical characteristics of the quantity Δx (h = 100 m) obtained by the method of statistical tests for N = 150 are equal to $M[\Delta x] = 100$ m; $\sigma^2[\Delta x] = 6.2 \cdot 10^6$ m^2. The numerical values of the moment of the first $Z^{(1)}$ and second $Z^{(2)}$ order computed by Eq. (6.70) are given in Table 6.6. Also presented there are the numerical values of the coefficients a_0, a_i, and a_{ii}.

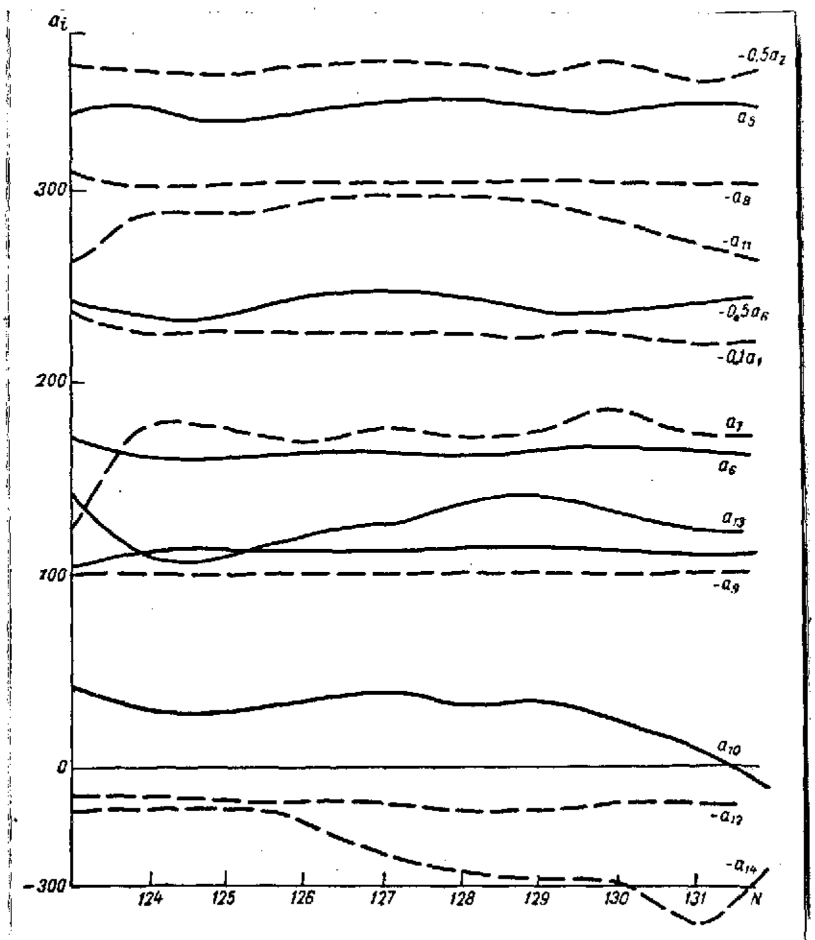
Table 6.6

j	i: 1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1032	−506	−407	170	−910	−104	−431	745	−260	−8	200	7	−246	244
2		−296	617	396	53	50	−159	4	−130	441	120	83	−11	−75
3			340	225	305	91	290	−137	275	−15	−149	−30	561	53
4				416	−686	−164	11	38	−328	−77	−152	−25	−150	−29
5					960	179	79	−188	−245	−169	−112	−113	404	202
6						26	432	−112	−60	7	−257	5	−277	−72
7							306	237	190	136	−303	−253	−69	−128
8								204	−79	−176	−32	246	−139	−76
9									234	421	−110	397	81	71
10										566	−73	52	−99	65
11											−12	−115	−170	−71
12												236	168	−75
13													118	−68
14														−100
a_{ii}	466	−198	120	158	430	−37	103	52	−167	233	−56	68	−109	−100
a_i	2183	−714	1046	423	305	142	−174	−322	−107	−15	−261	−19	100	−48

Table 6.1 shows the convergence of the coefficients a_i (i = = 1, 2, ..., m) as a function of the number N defining the number of the sequence (5.78).

From an analysis of the results it follows that the convergence of the method of statistical tests, when estimates of the vector Z are being determined, is quite high for the essential random factors. Estimates of the vector Z converge somewhat worse for factors that have little effect on the scatter of the phase coordinate Δx.

The calculations were made on a M-220 digital computer and afford the conclusions that individual random factors affect the scatter of the coordinate Δx and the necessity of including them when carrying out investigations for a given class of processes. Table 6.7 gives the numerical values of the contributions made by the first four linear terms of the expansion, along with the numerical value of the dispersion of coordinate Δx in absolute values and in percentages.



The data of Tables 6.6 and 6.7 show that the first four random factors in a linear model determine with a precision up to 0.5 percent the dispersion of the coordinate Δx. The remaining random factors have virtually no affect on the descent of the flight vehicle described by Eqs. (6.1.1). /235

Fig. 6.1. Variation of coordinates in $a_i(t)$ ($i = 1, 2, \ldots, 14$) as a function of the numbers of sample elements.

Table 6.7

n	1	2	3	4
$a_i^2 \cdot 10^6$	4,75	0,49	1,09	0,176
$\frac{a_i^2}{\sigma^2[\Delta x]}$ %	76,6	7,9	17,6	2,8

CHAPTER SEVEN /236

NUMERICAL OPTIMIZATION OF CONTROL ALGORITHMS FOR FLIGHT VEHICLE MOTION

7.1. Optimization of Processes of Controlling Flight Vehicle Motion

The above-examined problems of the scatter of kinematic parameters of the trajectories of flight vehicles in the dense atmospheric layers are a constituent part of processes in the statistical optimization of control parameters of their motion.

Examples 5.1 and 5.2 examined linear control algorithms providing compensation for deviations of kinematic parameters of flight vehicle motion from their reference values. In principle, the algorithms can also be nonlinear functions of the mismatches of phase coordinates X. The problem of synthesizing the structure and the parameters of the algorithm of the $\Delta U(X)$ can be formulated as a variational problem /43, 59, 74/, of minimizing some specified quality criterion /19, 32, 33, 43, 67, 79/ when there are relations in the form of differential equations (5.2).

However, in most practical problems of the control of flight vehicle motion it is not possible to solve the problem of synthesizing the control algorithms (the problem of determining the optimal structure and parameters of control actions) by employing necessary and sufficient conditions for the optimality of functionals /74/ owing to the complexity or cumbersomeness of the known formalizations. Usually, by employing formalizations of variational methods we can only indicate the class of functions to which the structure of optimal control belongs.

Among the practical problems of control, by virtue of the specificity of control processes and actuating devices of control systems, or the state of technology in realizing optimal controls, the class of functions in which one seeks the optimal control is restricted. Therefore, one widely used approach to setting up control algorithms that are satisfactory in practice is the approach based on direct methods of solution [43, 59]. Essentially, the latter amounts to the following. A class of unknown functions /237 (dependences of control actions on mismatches of process phase states) wholly defined by a finite set of parameters (coefficients) is specified, and the numerical values of the parameters (coefficients) are calculated from the condition that an extremal value of the criterion of process control is ensured on an admissible set of the parameters sought for. Mathematically, these problems are formulated thusly.

Problem 7.1. It is required to find an extremum (for sake of definiteness, we will assume the minimum) of the quality criterion

$$\bar{I} = I(k_1, k_2, \ldots, k_s) \tag{7.1}$$

on an open set Ω_K of parameters k_1, k_2, ..., k_s, if the quality criterion (7.1) is computed for solutions to the system of differential equations

$$\begin{gathered} \dot{x}_i = f_i(x_1, x_2, \ldots, x_n, \Delta u_1, \Delta u_2, \ldots, \Delta u_r, \xi_1, \ldots, \xi_m), \\ x_i(t_0) = x_{i,0}, \ (i = 1,2, \ldots, n). \end{gathered} \tag{7.2}$$

and if the controls $\Delta u_1, \Delta u_2, \ldots, \Delta u_r$ are defined by the functions:

$$\Delta u_j = \Delta u_j (\Delta x_i (t), k_l; \ i = 1, 2, \ldots, n; \ l = 1,2, \ldots, s). \tag{7.3}$$

In the formulated problem 7.1, the quality criterion is usually specified in the form

$$\begin{gathered} I(k_1, k_2, \ldots, k_s) = M [\Phi (\Delta x_i (t), \Delta u_j, k_l; \ i = 1,2, \ldots, n; \\ j = 1,2, \ldots, r; \ l = 1,2, \ldots, s], \end{gathered} \tag{7.4}$$

since the solutions to Eqs. (7.2) are stochastic by virtue of the randomness of perturbations $\xi_1(t), \xi_2(t), \ldots, \xi_m(t)$.

Problem 7.2. It is required to find the extremum of the quality criterion (7.1) on a closed set Ω_K of parameters k_1, k_2, ..., k_s if the quality criterion is computed by Eq. (7.4) for solutions to the system of nonlinear differential equations (7.2), and if the controls are specified in the form (7.3).

The set Ω_K is usually assigned as the intersection of two sets $\Omega_K^{(1)}$ and $\Omega_K^{(2)}$ defined as follows. The set $\Omega_K^{(1)}$ is a set of parameters $k_1, k_2, \ldots, k_s$ defined by constraints of the form:

$$\check{K} \leqslant K \leqslant \hat{K}, \tag{7.5}$$

where $K = \{k_1, k_2, \ldots, k_s\}$ is the s-dimensional vector of the (optimized) parameters sought for; and $\check{K}$, $\hat{K}$ are vectors of the numerical values defining the range of variation of the parameters being optimized.

The set $\Omega_K^{(2)}$ is specified by constraints of the form:

$$Q(k_1, k_2, \ldots, k_s) \leqslant Q^0, \tag{7.6}$$

where Q^0 is the 1-dimensional vector of the assigned numerical value; Q is the 1-dimensional vector of the assigned functions of the parameters being optimized.

The functions $Q_j(k_1;\ 1 = 1, 2, \ldots, s)$ are computed for the solutions to the system of differential equations (7.2) and are specified in the form /238

$$\overline{Q}(K) = M\,[G(\Delta x_i, \Delta u_j, k_l),\ i = 1,2,\ldots,n;\ j = 1, 2,\ldots, r; \\ l = 1, 2,\ldots, s]. \tag{7.7}$$

Thus, in problems of optimizing control systems of dynamic processes, the quality criterion (7.4) and the constraint (7.7) are computed in the general case for solutions of nonlinear stochastic differential equations, which naturally leads to the implicit dependence of the quality criterion (7.1) and the constraints (7.6) on the parameters (of vector K) being optimized.

Therefore it does not appear possible in advance to indicate or determine the class of functions to which these functions I(K) and Q(K) belong. One can only note that problem (7.1) is in the class of problems of searching for an extremum of an implicit function of many variables. Problem 7.2 is in the class of problems of nonlinear programming with a nonlinear dependence on the parameters being optimized of both the control criterion 7.1 as well as of constraints (7.6).

7.2. Methods of Searching for Extremum of Functions of Many Variables

There are a considerable number of methods, algorithms, and procedures for solving the problem of searching for an extremum of functions of many variables both with [37] and without allowance for constraints [61, 71]. Most of these are based on certain hypotheses about the structure of function I(K). One of the widely used is the hypothesis that states that the function being optimized is unimodal on the set of parameters Ω_K. Essentially, the assumption that the quality criterion I(K) is unimodal cannot be validated in problems 7.1 and 7.2 without a preliminary study of the function being optimized by computing the quality criterion at a number of points on the set Ω_K. The process of computing the quality criterion for a specified numerical value of vector K will be referred to as an experiment.

The methods of searching for an extremum of the quality criterion I(K) must take into account the absence of a priori information on the structure function under study, its unimodality, uniqueness, or polyextremality, and so on. Also, in algorithms for optimizing the quality criterion I(K), no small role is played by methods of computing the possible errors in carrying out the experiment (in computing the quality criterion by analytic methods or by numerical methods employing analog or digital computers).

Summing up the material on searching for an extremum of functions of many variables, we can present a classification, given in Fig. 7.1, that is convenient in setting forth the methods and algorithms of numerical optimization. Under this proposed classification, all methods of searching for an extremum of func- /239 tions of many variables can be divided into three large groups:

1) methods of random search;

2) determinate iterative methods of search; and

3) combined methods merging determinate methods with random search.

As applied to the present class of problems under study, methods of random search are most preferable by virtue of the absence of a priori information about the function being optimized. However, the fairly large number of experiments by which the extremum of the function optimized is attained casts doubt on the applicability of these methods in practical problems of optimizing control processes.

Determinate iterative methods permit reducing to the minimum the number of experiments when searching for an extremum of a specified function. The absence of a priori information about the mono-extremality of the function under study does not enable us to state that the optimum found is absolute.

The distinguishing feature of determinate iterative methods is the fact that in searching for a local extremum of function $I(K)$, one selects the initial point $K = K^{(0)}$ in the space of parameters Ω_K. Different methods of studying the behavior of the function $I(K)$ in the neighborhood of point $K = K^{(0)}$ and the depth of this search also essentially underly all methods and algorithms of determinate search. These features also are the basis for the classification of determinate methods (Fig. 7.1).

To clarify the problem of the uniqueness of the extremum of a function $I(K)$ on the set of parameters Ω_K found by means of determinate methods of search, we can use two types of algorithms. One is associated with the determinate subdivision of the set Ω_K into the subsets $\omega^{(j)}$, $(j = 1, 2, \ldots, N)$ belonging to the set Ω_K, and the search of the extremum in each of the subsets. The second approach can be associated with combining the methods of determinate search and random selection of the initial point $K = K^{(0)}$, with the setting up of bounds to the subset of parameters $\omega^{(j)}$ under study in order to exclude it from further investigation.

Among the determinate methods of search we can single out four groups of methods differing in the depth of the search of the structure of the quality criterion $I(K)$:

screening methods;

Gauss-Seidel method;

methods based on the local representation of the quality criterion by approximating hypersurfaces of specified order; and

methods associated with approximating the quality criterion on a preselected subset of parameters being optimized by approximating hypersurfaces of specified order.

Screening methods involve singling out on a set Ω_K the /241
parameters of a series of points at which an experiment is conducted. Comparing the numerical values of the quality criterion

Fig. 7.1. Classification of methods of searching for extremum functions of many variables.

[Key to Fig. 7.1 on preceding page]

1. Methods of searching for extremum of functions of many variables
2. Random search
3. Determinate methods of search
4. Combination methods of search
5. Method of screening
6. Gauss-Seidel method
7. Methods of local approximation
8. Method of approximating polynomials
9. Determinate orientation with random search along it
10. Methods of search for extremum of function of one variable
11. Algorithms without "memory"
12. Algorithms with "memory
13. Random orientation with determinate search along it
14. Methods based on constructing a hyperplane
15. Methods based on constructing second-order hypersurfaces
16. Methods based on constructing hypersurfaces of order higher than two
17. Random search with successive approximation of results by hypersurfaces
18. Method of steepest descent
19. Method of gradient
20. Method of differential equations

I(K) in each of the experiments makes it possible to single out at least a subset of parameters where the extremum sought for is found. This statement is evidently valid for the class of unimodal functions. The effectiveness of screening methods for the case of an arbitrary class of a function being optimized is considerably reduced, since in this case we can significantly increase the number of nodes on the subset Ω_K of parameters at which it is necessary to conduct an experiment. Since the bounds of the set Ω_K in problem 7.1 usually are not exactly defined, screening methods involve a considerable number of experiments and their present application is scarcely advisable. It should be noted that in some polyextremal problems in which extrema of identical values of I(K) are present, screening methods can be the only methods of determining all extrema.

The Gauss-Seidel method involves investigating the quality criterion in the plane of a single parameter k_i for fixed numerical values of the remaining parameters of the elements of vector K. The advantage of the Gauss-Seidel method over the screening methods lies in the investigation of the structure of the quality criterion in the plane of a single parameter, since it reduces the process of multiparametric optimization to uniparametric, that is, to searching for an extremum of the function of a single variable, and involves two algorithms.

The first of these requires knowledge of the range of variation of each parameter being optimized, that is, the bounds of set $\Omega_K^{(i)}$. Here, for a specified set $\Omega_K^{(i)}$ we investigate by means of the screening method a cross section of the quality criterion: we find that the nature of the function in this cross section, determine the number of extrema, and so on. In the second case, we use search algorithms from the initial point $k_i^{(0)}$. A specific local extremum is employed as a first approximation in the parameter k_i when investigating the function for the other parameters.

Usually, the Gauss-Seidel method is used when there is a small number of parameters being optimized ($s < 5$). The applicability of the Gauss-Seidel method for a larger number of parameters involves a considerable number of experiments and when the cost of each experiment is high can scarcely be justified.

Considerable acceptance as search methods has been gained by algorithms involving a local approximation of the function under study with hypersurfaces of specified order in a small neighborhood selected as the initial approximation of the vector $K = K^{(0)}$. Essentially, this group of methods and algorithms involves studying the structure of the quality criterion only in the ϵ-neighborhood of the initial approximation of vector $K^{(0)}$. In turn, the difference between algorithms and method involving local approximation of the quality criterion lies in the depth to which the behavior of the quality criterion is studied in the ϵ-neighborhood of the space of parameters of $K^{(0)}$. We can single out three groups of algorithms that share common approaches to the search process: /242

1) methods of constructing a hyperplane in the space of parameters Ω_K passing through the point $K^{(0)}$;

2) methods of constructing a hypersurface of second order passing through the point $K^{(0)}$; and

3) methods of constructing a hypersurface higher than the second order passing through the point $K^{(0)}$.

The expediency of this subdivision involves algorithms for employing the information obtained on the behavior of the optimized function in the neighborhood of point $K^{(0)}$. This group of methods shares the fact that at the point $K = K^{(0)}$ a Taylor series of the form

$$\hat{I}(K^{(0)} + \Delta K) = I(K^{(0)}) \sum_{i=1}^{s} \frac{\partial I}{\partial k_i}(K^{(0)}) \Delta k_i + \\ + \frac{1}{2} \sum_{i,j=1}^{s} \frac{\partial^2 I(K^{(0)})}{\partial k_i \partial k_j} \Delta k_i \Delta k_j + \ldots + \frac{1}{l!} \sum_{i,j,\ldots,\mu=1}^{s} \frac{\partial^l I(K^{(0)})}{\partial k_i \partial k_j \ldots \partial k_l} \times \\ \times \Delta k_i \Delta k_j \ldots \Delta k_l, \tag{7.8}$$

where

$$\Delta k_i = k_i - k_i^{(0)}.$$

can be constructed.

Obviously, constructing the series (7.8) when it has a fairly large number of terms involves a large number of experiments or cumbersome calculations in computing the partial derivatives:

$$\frac{\partial I}{\partial k_i}, \frac{\partial^2 I}{\partial k_i \partial k_j}, \frac{\partial^3 I}{\partial k_i \partial k_j \partial k_\nu}, \ldots \quad (i \leqslant j \leqslant \nu \leqslant \ldots = 1,2, \ldots, s).$$

Use of the linear portion of series (7.8) in the form

$$\hat{I}(K^{(0)}+\Delta K)=I(K^{(0)})+\sum_{i=1}^{s}\frac{\partial I}{\partial k_i}(K^{(0)})\,\Delta k_i \tag{7.9}$$

enables us to establish only the direction in the space of parameters Ω_K in which the quality criterion I(K) is decreased. To search for an extremum in this direction we can use familiar methods and algorithms (Fig. 7.2):

the method of steepest descent; and

the method of principal components of the gradient, and other methods that reduce the problem of the multiparametric approach to a problem of uniparametric search for an extremum of functions along the direction of the gradient of the quality criterion at at the point $K = K^{(0)}$.

Use of the quadratic model /36/ for the criterion, when there /244
are three terms in the series (7.8):

$$\hat{I}(K^{(0)}+\Delta K)=I(K^{(0)})+\sum_{i=1}^{s}\frac{\partial I}{\partial k_i}(K^{(0)})\,(k_i-k_i^{(0)})+ \\ +\frac{1}{2}\sum_{i,j=1}^{s}\frac{\partial^2 I(K^{(0)})}{\partial k_i\partial k_j}(k_i-k_i^{(0)})(k_j-k_j^{(0)}). \tag{7.10}$$

solves the problem of searching for the gradient and determining the size of the step in the parameter space.

Necessary conditions for the minimum of Eq. (7.10) yield a system of linear algebraic equations for determining the magnitude of the new approximation of vector K in the parameter space Ω_K:

$$\frac{\partial I}{\partial k_i}(K^{(0)})+\sum_{j=1}^{s}\frac{\partial^2 I}{\partial k_i\partial k_j}(K^{(0)})\,(k_j-k_j^{(0)})=0. \tag{7.11}$$

And, finally, nonlinear models of the quality criterion in which the number of terms in the series (7.8) is larger than three yield a fuller representation of the behavior of function I(K) in the ϵ-neighborhood of the point $K = K^{(0)}$, however their use does not afford as effective an algorithm for searching for the new approximation to the vector K in the parameter space Ω_K as the second-degree expansion (7.10). To search for the vector of parameters in the next approximation it is required to use either the method of differential equations, or methods of searching for

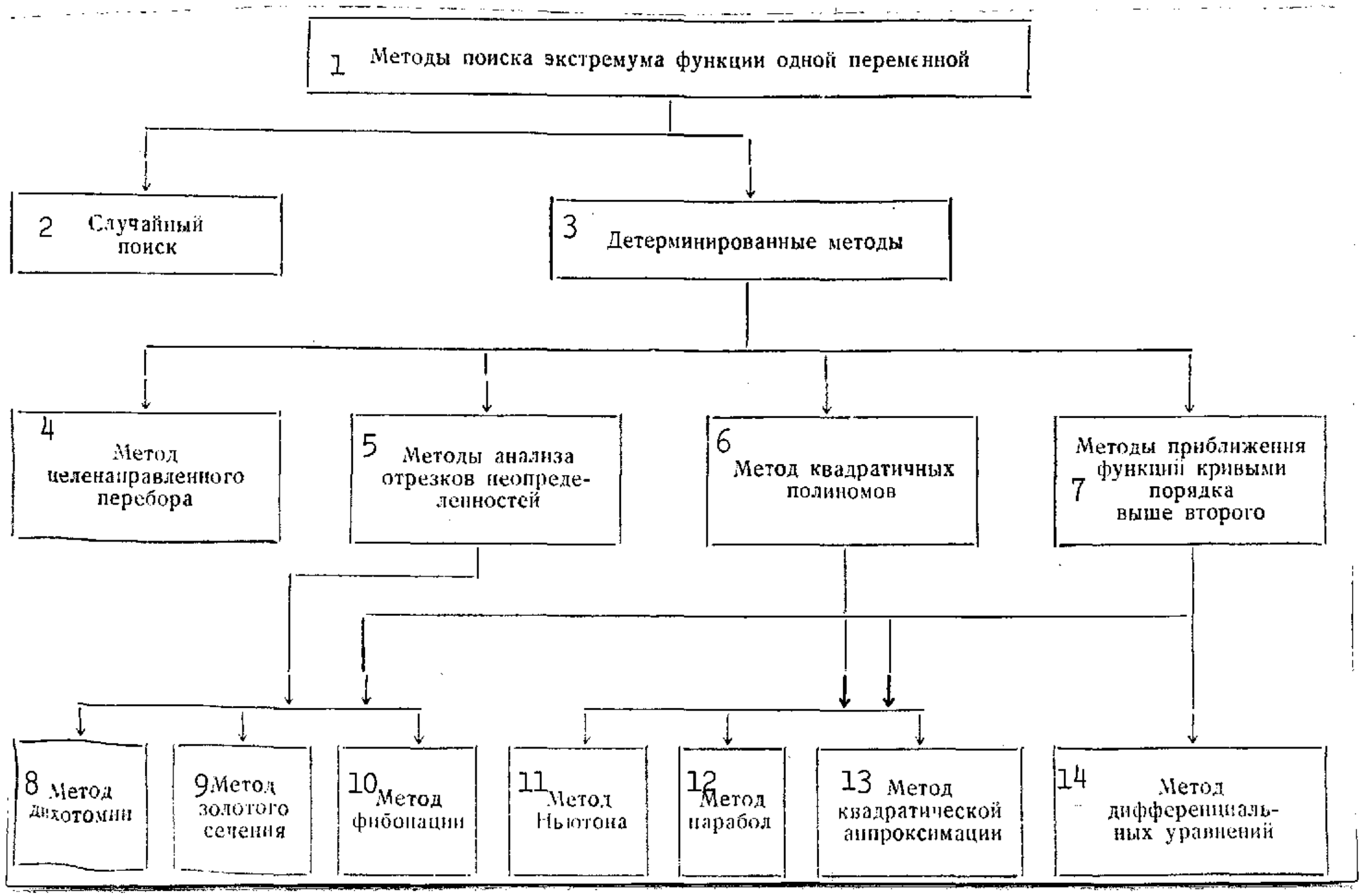


Fig. 7.2. Classification of methods of searching for extremum of functions of a single variable.

/Key on following page./

/243

/Key to Fig. 7.2 on preceding page./

1. Methods of search for extremum of functions of one variable
2. Random search
3. Determinate methods
4. Method of purposeful screening
5. Methods of analyzing segments of indeterminacies
6. Methods of quadratic polynomials
7. Methods of approximating functions by curves with order higher than two
8. Method of dichotomy
9. Method of golden mean
10. Method of Fibonacci
11. Newton's method
12. Method of parabolas
13. Method of quadratic approximation
14. Method of differential equations

an extremum of a function of parameters being optimized, assigned in explicit form. The method of differential equations essentially is a method of steepest descent with a small step and consists of searching for a steady function described by the differential equations

$$\frac{dk_i}{dt} = \lambda_i \frac{\partial \hat{I}}{\partial k_i} \quad (i = 1, 2, \ldots, s), \tag{7.12}$$

where $\hat{I}$ is the model of the quality criterion and λ_i are the coefficients of proportionality.

Solution of equation (7.12) on a digital computer does not pose serious difficulties, therefore in principle nonlinear models higher than the second degree can be used in searching for an extremum of a function of many variables if their construction is possible.

We must at once note that success in the procedure of searching for an extremum of functions of many variables using local nonlinear models depends to a large extent on the cleanness with which experiments are carried out in determining the partial derivatives of the expansion (7.8).

A second feature in the application of methods of local approximation is the requirement that the function I(K) be unimodal, since the presence even of minor high-frequency oscillatory components /245 in the function being optimized leads to instability of the extremum search algorithm.

We can sidestep the disadvantages of search methods involving local approximation of a function by employing a group of methods involving approximation of the function I(K) on a specified set $\omega(j)$.

A difference between the methods of searching involving approximating the criterion on a set $\omega^{(j)}$ from the method of searching involving local approximation of the criterion in the ϵ-neighborhood of the point $K^{(0)}$ is the fact that in this case, instead of the expansion (7.8) of function I(K) in the neighborhood of point $K^{(0)}$, we set up a problem of the optimal approximation of function I(K) by a polynomial of specified degree

$$\hat{I} = a_0 + \sum_{i=1}^{s} a_i \Delta k_i + \sum_{\substack{i, j=1 \\ (i \leqslant j)}}^{s} a_{ij} \Delta k_i \Delta k_j \tag{7.13}$$

from the condition that the smallest value is ensured, for example, the integral quadratic error of a similar representation

$$J = \int_{\omega(j)} \varepsilon^2 (\Delta K) \, d\Delta K, \tag{7.14}$$

where

$$\varepsilon (\Delta K) = I (\Delta K) - \hat{I} (\Delta K). \tag{7.15}$$

As to the remaining features, methods of solving the problem of searching for an extremum of a function I(K) based on its model $\hat{I}(K)$ are analogous to those presented above when local approximations with linear and nonlinear models of the quality criterion are used.

The problem of the necessity of the singling out methods of approximating the function $\hat{I}(K)$ by the polynomial $\hat{I}(K)$ on the set $\omega^{(j)}$ into a separate group of methods can arise. It is difficult to respond uniquely to this question. On the one hand, computational aspects of this group of methods differ appreciably from the computational aspects of local-approximation methods. In the first case, in constructing the series (7.8) we can use either functions of the sensitivity of the quality criterion with respect to the parameters being optimized [50], or formulas of differences for approximate computation of partial derivatives. In the second case, the principal apparatus for constructing the optimizing polynomials is the method of least squares (integral or pointwise), solving both the problem of computing the required regression coefficients of polynomials (7.13), as well as the problem of smoothing the "irregularities" of the function I(K).

In addition to this difference, methods of approximating polynomials enable us to use to its full extrent the "prehistory" of the extremum search process, which cannot be said of local-approximation methods. These algorithms cannot be employed in local-approximation methods. The differences listed are fairly essential in order to single out the method of approximating /246
polynomials into a separate group of extremum search methods.

On the other hand, in either case we are dealing with the approximate representation of the quality criterion by Taylor series (7.8) or by approximating polynomial (7.13) which do not differ in form from each other. This form similarity of representation (7.8) and (7.13) differing in content can raise an objection when the methods of the approximating polynomials are placed in a separate group of methods. Since the presentation of methods and algorithms under this classification is more preferable, we can adopt the solution of extending to the methods of approximating polynomials autonomy in the classification of methods of searching for an extremum of functions of many variables.

7.3. Methods of Computing Partial Derivatives for Statistical Characteristics of Stochastic Processes

When stochastic processes described by the following nonlinear equations are optimized:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, t, k_1, k_2, \ldots, k_s, v_1, v_2, \ldots, v_m),$$
$$x_i(t_0) = x_{i,0} \quad (i = 1,2,\ldots,n),\ t \in [t_0, T], \tag{7.16}$$

one of the serious problems of a computational nature is the problem of computing partial derivatives of Taylor series (7.8) for the criterion I(K) being optimized.

Generally quality criteria can be of two types

$$I_1 = M\left[\int_0^T \Phi(x_i, t)\,dt\right], \tag{7.17}$$

$$I_2 = M[\Phi(x_i, T)]. \tag{7.18}$$

Let us determine the partial derivatives of criteria (7.17) and (7.18) based on control parameters $k_1, k_2, \ldots, k_s$. We will have

$$\frac{\partial I_1}{\partial k_l} = M\left[\int_0^T \left\{\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}\frac{\partial x_i}{\partial k_l} + \frac{\partial \Phi}{\partial k_l}\right\} dt\right]; \tag{7.19}$$

$$\frac{\partial I_2}{\partial k_l} = M\left[\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}\frac{\partial x_i}{\partial k_l} + \frac{\partial \Phi}{\partial k_l}\right] \quad (l = 1,2,\ldots, s); \tag{7.20}$$

$$\frac{\partial^2 I_1}{\partial k_l \partial k_\nu} = M\left[\int_0^T \left\{\sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial k_l}\frac{\partial x_j}{\partial k_\nu} + \frac{\partial^2 \Phi}{\partial k_l \partial k_\nu} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}\frac{\partial^2 x_i}{\partial k_l \partial k_\nu}\right\} dt\right], \tag{7.21}$$

$$\frac{\partial^2 I_2}{\partial k_l \partial k_\nu} = M\left[\sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial k_l}\frac{\partial x_j}{\partial k_\nu} + \frac{\partial^2 \Phi}{\partial k_l \partial k_\nu} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}\frac{\partial^2 x_i}{\partial k_l \partial k_\nu}\right] \quad (l \leqslant \nu = 1,2,\ldots, s), \tag{7.22}$$

/247

where $\left[\frac{\partial x_i}{\partial k_l}, \frac{\partial^2 x_i}{\partial k_l \partial k_\nu}\right]$ are the partial derivatives of the solutions to nonlinear stochastic equations (7.16).

To compute the partial derivatives of the first order $\left[\frac{\partial x_i}{\partial k_l}\right]$, second order $\left[\frac{\partial^2 x_i}{\partial k_l \partial k_\nu}\right]$, and higher orders for solutions to nonlinear equation (7.16), we can make use of the differential equations of sensitivity. Thus, the system of differential equations of sensitivity of first order for Eqs. (7.16) is of the form

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial k_l}\right) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\frac{\partial x_j}{\partial k_l} + \frac{\partial f_i}{\partial k_l},$$

$$\frac{\partial x_i}{\partial k_l}(t_0) = \left(\frac{\partial x_i}{\partial k_l}\right)_0 \quad (i = 1,2,\ldots, n),\ (l = 1,2,\ldots, s). \tag{7.23}$$

The second-order partial derivatives for solutions of non-linear equations can be computed using the following system of differential equations

$$\frac{d}{dt}\left(\frac{\partial^2 x_i(t)}{\partial k_l \partial k_\nu}\right) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial^2 x_j}{\partial k_l \partial k_\nu} + \frac{\partial^2 f_i}{\partial k_l \partial k_\nu} + \\ + \sum_{p,\, j=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_p} \frac{\partial x_j}{\partial k_l} \frac{\partial x_p}{\partial k_\nu} \qquad (l, \nu = 1,2,\ldots, s),\ (i = 1,2,\ \ldots,\ n). \tag{7.24}$$

Systems of differential equations for determining partial derivatives of higher order can be set up based on an algorithm that is analogous to the one described above.

Introducing the notation

$$w_l^i(t) = \frac{\partial x_i(t)}{\partial k_l} \quad (i = 1,2,\ldots, n;\ l = 1,2,\ldots, s);$$

$$w_{pl}^i(t) = \frac{\partial^2 x_i(t)}{\partial k_p \partial k_l} \quad (p,\, l = 1,2,\ldots, s;\ i = 1,2,\ldots, n), \tag{7.25}$$

systems of equations (7.23) and (7.24) can be represented in the following form: /248

$$\dot{w}_l^i(t) = \sum_{j=1}^{n} a_{ij}(t)\, w_l^j + \xi_l^i(t) \qquad (i = 1,2,\ldots, n),\ (l = 1,2,\ \ldots,\ s),$$

$$\dot{w}_{l\nu}^i(t) = \sum_{j=1}^{n} a_{ij} w_{l\nu}^j + \sum_{p,\, j=1}^{n} b_{jp}^i w_l^j w_\nu^p + \xi_{l\nu}^i(t) \quad (i = 1,2,\ldots, n; \\ l, \nu = 1,2\ldots, s), \tag{7.26}$$

where we use the notation

$$a_{ij} = \frac{\partial f_i}{\partial x_j},\ b_{jp}^i = \frac{\partial^2 f_i}{\partial x_j \partial x_p},$$

$$\xi_l^i = \frac{\partial f_i}{\partial k_l},\ \xi_{l\nu}^i = \frac{\partial^2 f_i}{\partial k_l \partial k_\nu}. \tag{7.27}$$

If the initial systems of equations (7.16) do not depend on the parameters k_1, k_2, ..., k_s, the initial conditions of systems of equations (7.26) are zero:

$$w_l^i(t_0) = w_{l\nu}^i(t_0) = 0 \quad (i = 1,2,\ldots, n),\ (l, \nu = 1,2,\ldots, s).$$

Eqs. (7.27) are random functions of the perturbing actions, therefore systems of equations (7.26) constitute a system of stochastic equations.

Substituting the notation (7.25) into Eqs. (7.19) - (7.22), we get

$$
\begin{aligned}
\frac{\partial I_1}{\partial k_l} &= M\left[\int_0^T \left\{\sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} w_l^i(t) + \frac{\partial \Phi(t)}{\partial k_l}\right\} dt\right] \quad (l = 1,2,\ldots, s);\\
\frac{\partial I_2}{\partial k_l} &= M\left[\sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} w_l^i(T) + \frac{\partial \Phi(T)}{\partial k_l}\right] \quad (l = 1,2,\ldots, s);\\
\frac{\partial^2 I_1}{\partial k_l \partial k_\nu} &= M\left[\int_0^T \left\{\sum_{i,j=1}^n \frac{\partial^2 \Phi(t)}{\partial x_i \partial x_j} w_l^i(t)\, w_\nu^j(t) + \frac{\partial^2 \Phi(t)}{\partial k_l \partial k_\nu} + \right.\right.\\
&\qquad \left.\left. + \sum_{i=1}^n \frac{\partial \Phi(t)}{\partial x_i} w_{l\nu}^i(t)\right\}\right] \quad (l, \nu = 1,2,\ldots, s);\\
\frac{\partial^2 I_2}{\partial k_l \partial k_\nu} &= M\left[\sum_{i,j=1}^n \frac{\partial^2 \Phi(T)}{\partial x_i \partial x_j} w_l^i(T)\, w_\nu^j(T) + \frac{\partial^2 \Phi}{\partial k_l \partial k_\nu} + \right.\\
&\qquad \left. + \sum_{i=1}^n \frac{\partial \Phi(T)}{\partial x_i} w_{l\nu}^i(T)\right] \quad (l, \nu = 1,2,\ldots, s).
\end{aligned}
\tag{7.28}
$$

By integrating system of differential equations (7.16) jointly with systems of differential sensitivity equations (7.26) for the specified realizations of the vector of random factors, we can obtain the realizations of solutions x_i (t, V) and the realizations of sensitivity functions w^i_{l} (t, V) and $w^i_{l\nu}$ (t, V). /249

Treating the necessary number of realizations of these functions by one of the methods of the statistical analysis of nonlinear systems of equations, by Eq. (7.28) we can compute the required partial derivatives of the mathematical expectations of the function Φ under study or the integral of it to set up Taylor series (7.8),

As a whole, use of differential sensitivity equations is quite an effective procedure, however several difficulties can crop up in its realization.

First of all, to get the differential sensitivity equations we must do a great deal of preliminary work. Secondly, differential sensitivity equations have a high system order.

Tables 7.1 and 7.2 give the numerical values of the order of the system N^r for sensitivity models of the first and second order, respectively, as a function of the order of the initial system of differential equations "n" and the order "s" of the vector of the parameters K being optimized.

Table 7.1
Order of System of Differential Equations for Computing First-Order Partial Derivatives

n	s: 1	5	10	15	20
1	2	6	11	16	21
5	10	30	55	80	105
10	20	60	110	160	210
15	30	90	165	240	315
20	40	120	220	320	420

Table 7.2
Order of System of Differential Equations for Computing Second-Order Partial Derivatives

n	s: 1	5	10	15	20
1	4	66	409	1 271	2 911
5	20	330	2 045	6 355	14 555
10	40	660	4 090	12 710	29 110
15	60	990	6 135	19 065	43 665
20	80	1 320	8 180	25 420	58 220

Table 7.3
Formulas for Calculating the Order of Differential Equations for Partial Derivatives of Solutions

d	1	2	3
N^r	$n(s+1)$	$\frac{n(s^2+3s+2)}{2}$	$\frac{n(2s^3+s^2+13s+6)}{6}$

The numerical values in Tables 7.2 and 7.2 for illustration of the high order of the system of differential equations were computed by the formulas given in Table 7.3. Also presented there is the formula for the computation of the order of the system of differential equations for the third-order partial derivatives. /250

Naturally, when the initial system has a high order ($n > 5$) of Eqs. (7.16) and when there is a considerable order of the vector of the parameters being optimized ($s > 5$), the order of the system of differential sensitivity equations for third-order models is an impressive figure ($N^r > 330$). From the foregoing it becomes obvious that integrating third-order sensitivity equations represents serious difficulties even for modern digital computers. The situation becomes somewhat better with sensitivity models of the second order (see Table 7.2). However, even here already when $s > 10$ and $n > 5$, the order of the system of equations that must be integrated to arrive at sensitivity functions of the first and second order also exceeds the number $N^r = 330$.

An applicable order of the system of differential equations is obtained only for first-order sensitivity models. Therefore to set up models of the quality criteria of the forms (7.17) and

(7.18), we can use the linear model of the expansions of solutions to nonlinear equation (7.16) in a Taylor series in increments of the parameters $\Delta k_1, \Delta k_2, \ldots, \Delta k_3$: being optimized:

$$x_i(t, K^{(0)} + \Delta K) = x_i(t, K^{(0)}) + \sum_{l=1}^{s} w_i^l(t, K^{(0)})\Delta k_l, \tag{7.29}$$

where

$$\Delta k_l = k_l - k_l^{(0)}, \quad (l = 1, 2, \ldots, s).$$

Substituting the linear model of solution (7.29) into the expressions for the quality criteria (7.17) and (7.18), we get

$$\begin{aligned} I_1 &= M\left[\int_0^T \Phi\left\{x_i(t, K^{(0)}) + \sum_{l=1}^{s} w_i^l(t, K^{(0)})\Delta k_l\right\} dt\right], \\ I_2 &= M\left[\Phi\left\{x_i(t, K^{(0)}) + \sum_{l=1}^{s} w_i^l(t, K^{(0)})\Delta k_l\right\}\right]. \end{aligned} \tag{7.30}$$

Using the Taylor series expansion of Eq. (7.30) in increments of the parameters Δk_l (l = 1, 2, ..., s), we can obtain an approximate expression for Taylor series (7.8).

Example 7.1. To illustrate the foregoing, let us look at the process of arriving at expansion (7.8) using a linear model of solutions for the quality criterion of the form

$$I_1 = MI\int_0^T \{X^*\overline{Q}X + U^*\overline{C}U\}\, dt \tag{7.31}$$

when we are dealing with the linear control

U = K*X (K is a column vector). (7.32)

Substituting Eq. (7.32) into criterion (7.31), we get /251

$$I_1 = M\left[\int_0^T X^*\overline{D}X dt\right], \tag{7.33}$$

where

$$\overline{D} = \overline{Q} + K\overline{C}K^*.$$

Since

$$X(t, \Delta K) = X(t, K^{(0)}) + w(t, K^{(0)})\Delta K,$$
$$K = K^{(0)} + \Delta K,$$

then we will have

$$I_1 = M\left[\int_0^T \{(X(t, K^{(0)}) + w(t, K^{(0)})\Delta K)^* (D^{(0)} + 2\Delta K \bar{C} K^{(0)} + \right.$$
$$\left. + \Delta K \bar{C} \Delta K^*)(X(t, K^{(0)}) + w(t, K^{(0)})\Delta K)\} dt\right]. \tag{7.34}$$

We can easily see that after all transformations are carried out, under the sign of the integral we get a fourth-degree polynomial in elements of vector ΔK. This procedure of setting up local expansions of mathematical expectations of several functions Φ in a Taylor series has been successfully used in a number of studies [36, 62] in optimizing linear and nonlinear stochastic control processes.

The idea of using a linear model of expansions of solutions to nonlinear equations (7.16) in a Taylor series is quite productive, since it enables us to compute only the differential sensitivity equations of first-order w_ℓ^1 (t) necessary in setting up the series (7.8), which can considerably cut down on the volume of preliminary work in arriving at sensitivity equations (7.26) and the volume of computations. Let us make an estimate of how effective this idea is for the case of setting up a quadratic model of the quality criterion. When Eqs. (7.28) are used, we must investigate $\frac{n}{2}(s^2+2s+2)$ equation; in setting up linear model (7.29), the number of required equations reduces to n (s + 1), that is, the number of required equations is reduced by a factor of $\eta_r = \frac{s+2}{2}$.

However, under this approach the function Φ must be a function higher than the first order in phase coordinates of the process under study, described by Eqs. (7.16).

Above we considered an exact method of computing the partial derivatives of expansion (7.8). The desired first-order partial derivatives can be computed also by employing approximate difference formulas, for example,

$$\frac{\partial I}{\partial k_i} \approx \frac{I(K^{(0)}, k_i^{(0)} + \Delta k_i) - I(K^{(0)})}{\Delta k_i} \quad (i = 1, 2, \ldots, s).$$

In the approximate computation of second-order partial derivatives we can use various formulas based on different experiment planning /42, 44/. As an example, let us consider the case of experiment planning according to the scheme shown in Fig. 7.3 a. Treatment of the results according to the scheme gives the following working formulas for computing partial derivatives /252

$$\left.\begin{aligned}
&\frac{\partial I}{\partial k_i} \approx \frac{I\left(k_i^{(0)} + \Delta\bar{k}_i\right) - I\left(k_i^{(0)} - \Delta\bar{k}_i\right)}{2\Delta k_i},\\
&\frac{\partial^2 I}{\partial k_i^2} \approx \frac{I\left(k_i^{(0)} + \Delta\bar{k}_i\right) + I\left(k_i^{(0)} - \Delta\bar{k}_i\right) - 2I\left(k_i^{(0)}\right)}{2\Delta\bar{k}_i^2},\\
&\frac{\partial^2 I}{\partial k_i \partial k_j} \approx \frac{\begin{array}{c} I\left(k_i^{(0)} + \Delta\bar{k}_i, k_j^{(0)} + \Delta\bar{k}_j\right) - I\left(k_i^{(0)} - \Delta\bar{k}_i\right) + \\ + I\left(k_j^{(0)} + \Delta\bar{k}_j\right) - 3I\left(k^{(0)}\right)\end{array}}{\Delta\bar{k}_i \Delta\bar{k}_j}\\
&(i < j = 1,2,\ldots, s).
\end{aligned}\right\} \tag{7.35}$$

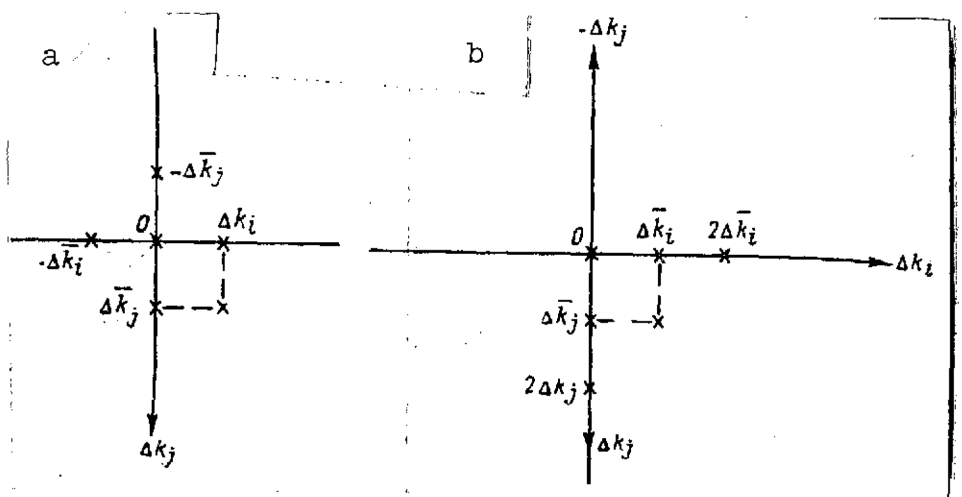


Fig. 7.3. Scheme of experiment planning for determining partial derivatives

At the present time, experiment planning for computing partial derivatives is carried out before beginning the computation process, that is, it is carried out rigorously and remains unchanged in the process of optimization as a function of the behavior of the function under study. This is basically caused by the desire to perform calculations of partial derivatives based on finite formulas.

On analogy with Fig. 7.3 a, we can also set up other schemes of experiment planning for computing approximate values of the partial derivatives of the quality criterion and the constraints.

(Fig. 7.3 b). For experiment planning represented in Fig. 7.3 b, the working formulas for computing partial derivatives of the first and second orders are of the form: /253

$$\frac{\partial I}{\partial k_i} \approx \frac{I\left(k_i^{(0)} + \Delta\bar{k}_i\right) - I\left(k_i^{(0)}\right)}{\Delta\bar{k}_i},$$

$$\frac{\partial^2 I}{\partial k_i^2} \simeq \frac{I\left(k_i^{(0)} + 2\Delta\bar{k}_i\right) - 2I\left(k_i^{(0)} + \Delta\bar{k}_i\right) + I\left(k_i^{(0)}\right)}{2\Delta\bar{k}_i^2},$$

$$\frac{\partial^2 I}{\partial k_i \partial k_j} \approx \frac{2I\left(k_i^{(0)} + \Delta\bar{k}_i,\ k_j^{(0)} + \Delta\bar{k}_j\right) - I\left(k_i^{(0)} + 2\Delta\bar{k}_i\right) - I\left(k_j^{(0)} + 2\Delta\bar{k}_j\right)}{2\Delta\bar{k}_i\Delta\bar{k}_j} \quad (i, j = 1,2,\ldots, s). \tag{7.36}$$

To solve the problem of computing estimates of partial derivatives, we can also use methods of optimal experiment planning /42, 77/ quite well elaborated for solving problems in multifactor analysis.

The minimum number of experiments needed to compute partial derivatives of the first and second orders for a quality criterion when difference formulas are used is

$$N^p = \frac{s^2 + 3s + 2}{2}$$

and is shown in the second row in Table 7.2.

In each of the experiments we must integrate only the initial system of differential equations (7.16) by as many times as is required by the procedure of computing the statistical characteristics of (7.17) or (7.18). The weak point of the difference scheme of computing partial derivatives is the indeterminacy of selecting the increments of coefficients $\Delta\bar{k}_i$ of the chosen initial approximation. Here, usually the function

$$\Delta\bar{k}_i \leqslant 0{,}1 k_i^{(0)}, \quad (i = 1,2,\ldots, s). \tag{7.37}$$

is employed.

Comparing these two approaches to solving the problem of arriving at local representations of the quality criterion in the neighborhood of the point $K^{(0)}$ in the form of a Taylor series, we can draw several conclusions.

When we are dealing with small numbers s and n ($s, n \leqslant 5$) for computing partial derivatives of the first and second orders, it is best to use the sensitivity equations, since when differential sensitivity equations are jointly integrated with the initial system of equations, owing to the common computational operations of numerical integration of the differential equations the time needed to arrive at the partial derivatives is less than when difference methods are used in solving this very same problem. Let us demonstrate this. The time required to solve this problem by difference methods can be determined by the formula

$$t_p = N^p t_1,$$

and the time required to solve the system of sensitivity equations is /254

$$t_r = \frac{1 + N^r n}{n+1} t_1 k_{op}.$$

In these formulas, we use the notation: t_1 is the time needed to solve the (n + 1)-th initial equation; N^p is the number of equations; n is the order of the initial system of equations without taking into account the differential equation for the independent variable (t = 1); and k_{op} is the coefficient of the common operations.

Let us compute the ratio

$$\eta_t = \frac{t_p}{t_r} = \frac{\eta'_t}{k_{op}} \quad \text{where} \quad \eta'_t = \frac{n+1}{n + \frac{1}{N^p}}.$$

The coefficient of the common operations k_{op} depends on the complexity of the right-hand sides of the sensitivity equations and usually varies in the range k_{op} = 0.6-1.1.

The numerical values for η_t and η'_t are in Table 7.4.

Table 7.4
Coefficient of Effectiveness for Difference Methods

s	N^p	η'_t				η_t			
		n							
		1	2	3	4	1	2	3	4
1	3	1.50	1.29	1.20	1.15	1.86	1.62	1.50	1.44
3	10	1.82	1.42	1.29	1.22	2.26	1.76	1.61	1.51
4	15	1.86	1.44	1.30	1.22	2.32	1.80	1.63	1.52
5	21	1.93	1.46	1.31	1.23	2.41	1.83	1.64	1.55
10	66	1.99	1.50	1.33	1.25	2.50	1.87	1.67	1.57

From Table 7.4 it follows that using the sensitivity models to determine the partial derivatives of expansion (7.8) is the more effective, the greater the number of parameters being optimized in the system under study and the smaller the order of the initial system of equations (7.16).

When selecting a method of computing partial derivatives of expansion (7.8), we must take into account both the advantages of the sensitivity models (in the sense of machine time outlays), as well as their drawbacks (considerable cumbersomeness in arriving at the sensitivity models), as well as the advantages of the difference methods associated with integrating only the initial system of differential equations and other criteria of practical importance.

In estimating the effectiveness of these two methods of arriving at expansion (7.8) from the standpoint of the cost of solving the problem posed, we can obtain a result that is contrary to that shown in Table 7.4.

For the very same process of optimization using the two above-described schemes of computing partial derivatives, the cost of the solution can be determined by the following relation: /255

$$C = \alpha_{prep} t_{prep} + \alpha_{prog} t_{prog} + \alpha_{c-o} t_{c-o} + \alpha_w t_s, \tag{7.38}$$

where α_{prep}, α_{prog}, α_{c-o}, are coefficients characterizing the cost of one hour of operation in preparation, programming, and check-out of the program on the digital computer; α_w is the cost of one hour's operation of the machine taking into account the cost of one hour of work by the operator; t_{prep} is the time required to prepare the problem for programming (hours); t_{prog} is the time for programming (hours); t_{c-o} is the time required for checking out the problem (hours); and t_s is the time required to solve the problem on the digital computer (hours).

Let us assume that the time of each operation is proportional to the number of the differential equations that are to be integrated in carrying out the optimization process. Table 7.5 gives all the formulas required for the calculations.

Referring to the data in Table 7.5, we can represent Eq. (7.38) for the two methods of computing partial derivatives in the following form:

Table 7.5
Formulas for Computing the Time of Preparation, Programming, Check-Out, and Solution of the Problem

Symbols	Methods of sensitivity theory	Difference methods
t_{prep}	$\beta_{prep}\left[\frac{2+n(s^2+3s+2)}{2}\right]$	$\beta_{prep}[(n+1)]$
t_{prog}	$\beta_{prog}\left[\frac{2+n(s^2+3s+2)}{2}\right]$	$\beta_{prog}[(n+1)]$
t_{c-o}	$\beta_{c-o}\left[\frac{2+n(s^2+3s+2)}{2}\right]$	$\beta_{c-o}[(n+1)]$
t_s	$\left[\frac{2+n(s^2+3s+2)}{2(n+1)}\right] t_1 n_{st} k_{op}$	$\left[\frac{s^2+3s+2}{2}\right] t_1 n_{st}$
γ	$\alpha_{prep}\beta_{prep} + \alpha_{prog}\beta_{prog} + \alpha_{c-o}\beta_{c-o}$	

Remark. n_{st} stands for the number of optimization stages.

$$C^r = \gamma\left[\frac{2+n(s^2+3s+2)}{2}\right] + \alpha_w t_1 k_{op} n_{st}\left[\frac{2+n(s^2+3s+2)}{2(n+1)}\right],$$

$$C^p = \gamma[(n+1)] + \alpha_w\left[\frac{s^2+3s+2}{2}\right] t_1 n_{st}.$$

For $\alpha_{prep} = 3$, $\alpha_{prog} = 1.5$, $\alpha_{c-o} = 35$, $\alpha_w = 32$, $\beta_{prep} = 3$, $\beta_{prog} = 0.5$, $\beta_{c-o} = 0.5$, $n_{st} = 5$, $t_1 = 0.1$, and $k_{op} = 0.8$, the numerical values of the criteria C^r and C^p as a function of the number n and s were computed and are in Table 7.6. Table 7.7 gives the ratio of the criteria C^r to C^p characterizing the effectiveness /256

$$\eta_C = \frac{C^r}{C^p}.$$

Table 7.6
Numerical Values of the Cost-Effectiveness Criterion

s	C^r				C^p			
	n							
	1	2	3	4	1	2	3	4
1	200	225	299	419	106	134	162	190
3	682	706	1134	1386	170	198	226	254
4	1026	1049	1422	2077	298	326	354	382
5	1438	1460	1984	2907	394	422	450	478
10	4534	4547	6196	9126	1114	1142	1170	1198

Table 7.7
Numerical Values of the Coefficient η_C

s	n			
	1	2	3	4
1	1,89	1,69	1,84	2,25
3	4,01	3,56	5,01	5,41
4	3,44	3,21	4,01	5,44
5	3,69	3,46	4,41	5,99
10	4,07	3,99	5,29	7,70

In Tables 7.8 and 7.9, analogous calculations for the very same coefficients α and β when $t_1 = 1$ are presented as an example.

Table 7.8
Numerical Values of the Effectiveness Criterion

s	C^r				C^p			
	n							
	1	2	3	4	1	2	3	4
1	365	490	600	676	535	562	588	615
3	1003	1470	1 860	2 132	1 655	1 682	1 708	1 735
4	1460	2170	2 760	3 172	2 455	2 482	2 508	2 535
5	2008	3060	3 840	4 420	3 415	3 442	3 468	3 495
10	6114	9310	11 940	13 780	10 615	10 642	10 668	10 695

Table 7.9
Numerical Values of the Coefficient η_C

s	n			
	1	2	3	4
1	0,683	0,870	1,010	1,100
3	0,610	0,875	1,090	1,230
4	0,594	0,873	1,100	1,260
5	0,585	0,891	1,120	1,270
10	0,565	0,878	1,120	1,290

From Table 7.7 it follows that applying difference formulas for computing partial derivatives is more effective from the standpoint of the cost criterion of the optimization process than using differential sensitivity equations for $t_1 = 0.1$ hour. /257

Here the coefficient of effectiveness η_C rises considerably with increase in the numerical values of s and n. The results in Tables 7.8 and 7.9 indicate the redistribution of the effectiveness of these methods discussed for different values of s and n. Here the coefficient of effectiveness η_C is smaller than unity for s and n and increases with increase of s and n.

Note that the data in Tables 7.6-7.9 were obtained for arbitrary values of the coefficients α and β. However, analysis of these results indicates the need for conducting guiding calculations when planning the optimization process for a digital computer, since by giving preference to a particular method of arriving at the Taylor series characteristics without analyzing the class of problems can lead to superfluous time outlays as well as to unnecessary cost in searching for an extremum of functions of many variables.

7.4. Methods of Constructing Approximating Polynomials

Underlying the methods of computing the coefficients of an approximating polynomial is the problem of the optimal approximation of a function of many variables with polynomial (7.13) having specified degree d. If the function I $(k_1, k_2, \ldots, k_s)$ is computed at the points $k_1^{( )}, k_2^{(i)}, \ldots, k_s^{(j)}$, $(j = 1, 2, \ldots, M)$, we have a problem of the optimal approximation of function $I(K)$ with the polynomial $I(\hat{K})$ at the points $k_1^{(j)}, k_2^{(j)}, \ldots, k_s^{(j)}$, that is, a problem of pointwise approximation /42/. Suppose the quality criterion of the approximation is of the form

$$J = \sum_{j=1}^{M} \left[I^{(j)}(K) - \hat{I}^{(j)}(K) \right]^2 \rho^{(j)}(K), \tag{7.39}$$

where $\rho^{(j)}(K^{(j)})$ is a weighting function characterizing the requirements on the precision of the approximation at the specified points $(j = 1, 2, \ldots, M)$.

When $\rho(K) = 1$, we have the method of least squares. Let us /258
look at the table of experiment planning for the case of constructing a quadratic approximating polynomial. For the set of coefficients $k_1^{(j)}, k_2^{(j)}, \ldots, k_s^{(j)}$ $(j = 1, 2, \ldots, M)$ suppose we have computed the function $I^{(j)}[\bar{K}^{(j)}] = I^{(j)}$.

The results of the calculations are in Table 7.10. Suppose $M > M_1 \left(M_1 = 1 + \frac{s^2 + s}{2} \right)$. Introducing for consideration the matrix X composed of elements from Table 7.10 of order (M, M_1) (columns from 1 to M_1, rows from 1 to M), and the matrix X*X of order (M_1, M_1), as well as a vector b of order (M, 1) (column $M_1 + 1$), we can write out the required ratio of the method of least squares for computing the coefficients of the approximating polynomial, which are elements of the vector A $(a_o, a_1, a_2, \ldots, a_s, a_{11}, a_{12}, \ldots, a_{ss})$. We will have

$$A = (X^*X)^{-1} X^* b. \tag{7.40}$$

Table 7.10
Table of Experiment Planning

No	1	2	3	...	s + 1	...	M_1	$M_1 + 1$
	0	k_1	k_2	...	k_s	...	k_s^2	$I^{(j)}$
1	1	$k_1^{(1)}$	$k_2^{(1)}$	...	$k_s^{(1)}$	...	$k_s^{2\,(1)}$	$I^{(1)}$
2	1	$k_1^{(2)}$	$k_2^{(2)}$	...	$k_s^{(2)}$	...	$k_s^{2\,(2)}$	$I^{(2)}$
3	1	$k_1^{(3)}$	$k_2^{(3)}$	...	$k_s^{(3)}$	...	$k_s^{2\,(3)}$	$I^{(3)}$
...	...	...	...	...	...	...	...	...
M	1	$k_1^{(M)}$	$k_2^{(M)}$	...	$k_s^{(M)}$	...	$k_s^{2\,(M)}$	$I^{(M)}$

From Eq. (7.40) it follows that by filling out Table 7.10 in conducting the experiments, we can easily compute the co-coefficients of the approximating polynomial based on the formula (7.40) presented if the matrix (X*X) is nonsingular. Since the number of terms of the quadratic polynomial is M_1, for the matrix (X*X) to be nonsingular it is necessary that the condition $M \geqslant M_1$ be satisfied, first of all, and that there be no repeated experiments, secondly.

When $\rho(K) \neq 1$, it is necessary that each j-th row of the planning matrix (all elements of the j-th row of Table 7.11) be multiplied by $\sqrt{\rho(\ )}$. Then we can use Eq. (7.40) to compute the elements of vector A.

Thus, the process of computing the coefficients of the approximating polynomial by the method of least squares involves the following:

1) carrying out $M \geqslant M_1$ experiments in computing the function $I\,[\bar{K}^{(j)}]$ and filling out Table 7.10;

2) transposing matrix X and multiplying the resulting matrix /259
by the matrix X;

3) inversion of matrix (X*X);

4) multiplying the resulting reciprocal matrix $(X^*X)^{-1}$ by the matrix X*; and

5) multiplying the matrix $(X^*X)^{-1}X^*$ by the vector b.

All computations in this program do not represent serious difficulties when the computations are conducted on a digital computer. Overall, the method of least squares admits of nonrigorous experiment planning, that is, selecting numerical values of the coefficients $k_i^{(j)}$ $(i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, M)$, in contrast to the difference formulas used in the preceding section. In rigorous experiment planning the matrix X must necessarily be specified before the beginning of the "experiments". Nonrigorous experiment planning can involve a random selection of the sequence of point $k_i^{(j)}$ $(i = 1, 2, \ldots, s)$, $(j = 1, 2, \ldots, M)$ in which the function $I^{(\ )}$ $[\bar{K}^{(j)}]$ is computed, or selecting the sequence of points $k_i^{(\ )}$ $(i = 1, 2, \ldots, s)$ on the condition that the sequence of numbers $I^{(j)}$ $(j = 1, 2, \ldots, M)$ at least not increase. This can be done by using various planning schemes.

7.5. Method of Stochastic Approximation in Problems of Constructing Approximating Polynomials

To construct models of the quality criterion and constraints, in the algorithms for optimizing control systems, we can use the method of stochastic approximation. This obviously becomes possible if we introduce a set ω of increments of the parameters ΔK being optimized and assume that on the set ω the elements of vector ΔK are random with an assigned symmetric function of probability density distribution $f\ (\Delta K)$. Then when the quality criterion $I\ (\Delta K)$ is approximated by polynomial $\hat{I}\ (\Delta K)$, with minimizing of the criterion for estimating the precision of the approximation

$$J = M\left\{\left[I(\Delta K) - \hat{I}(\Delta K)\right]^2\right\}$$

we can use the above-derived working formulas for computing the coefficients of the approximating polynomial. Thus, under the quadratic model of the quality criterion

$$\hat{I}(\Delta K) = a_0 + \sum_{i=1}^{s} a_i \Delta k_i + \sum_{\substack{i,\ j=1 \\ (i \leq j)}}^{s} a_{ij} \Delta k_i \Delta k_j$$

we will have the following working formulas for computing the coefficients a_0, a_i, and a_{ij}: /260

$$a_0 = \left(1 + \sum_{i=1}^{s} \frac{\left(M\left[\Delta k_i^2\right]\right)^2}{M\left[\Delta k_i^4\right] - \left(M\left[\Delta k_i^2\right]\right)^2}\right) Z_0 - \sum_{i=1}^{s} \frac{M\left[\Delta k_i^2\right] Z_{ii}^{(2)}}{M\left[\Delta k_i^4\right] - \left(M\left[\Delta k_i^2\right]\right)^2}; \tag{7.41}$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{M\left[\Delta k_i^2\right] M\left[\Delta k_j^2\right]} & (i < j = 1,2,\ldots, s); \\ \dfrac{Z_{ii}^{(2)} - M\left[\Delta k_i^2\right] Z_0}{M\left[\Delta k_i^4\right] - \left(M\left[\Delta k_i^2\right]\right)^2} & (i = j = 1,2, \ldots, s); \end{cases}$$

$$a_i = \frac{Z_i^{(1)}}{M\left[\Delta k_i^2\right]} \quad (i = 1,2, \ldots, s). \qquad \text{(7.41) (cont)}$$

where

$$Z_0 = M_{\Delta K}\left[I(\Delta K)\right];$$

$$Z_i^{(1)} = M_{\Delta K}\left[I(\Delta K)\Delta k_i\right] \quad (i = 1,2,\ldots, s);$$

$$Z_{ij}^{(2)} = M_{\Delta K}\left[I(\Delta K)\Delta k_i \Delta k_j\right] \quad (i \leqslant j = 1,2,\ldots, s). \qquad (7.42)$$

Eq. (7.41), under the normal law of probability density distribution of vector ΔK with assigned root mean square values of its elements σ_i ($i = 1, 2, \ldots, s$), are of the form

$$a_0 = \left(1 + \frac{s}{2}\right) Z_0 - \frac{1}{2} \sum_{i=1}^{s} \frac{Z_{ii}^{(2)}}{\sigma_i^2};$$

$$a_i = \frac{Z_i^{(1)}}{\sigma_i^2} \quad (i = 1,2, \ldots, s);$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{\sigma_i^2 \sigma_j^2} & (i < j = 1,2,\ldots, s); \\ \dfrac{Z_{ii}^{(2)} - \sigma_i^2 Z_0}{2\sigma_i^4} & (i = j = 1,2,\ldots, s) \end{cases} \qquad (7.43)$$

One feature of using Eq. (7.43) when computing the coefficients a_0, a_i, and a_{ij} is that on the set $\Omega_{\Delta K}$ a central point $K^{(0)}$ is singled out and all elements of the vector K are grouped relative to it with assigned probability distribution (we are talking about the vectors ΔK). The elements of vector ΔK can be distributed with equal probability relative to the point $K^{(0)}$ /261 on the set ω. If $\Delta k_i \in [-b_i, b_i]$, then when $M[\Delta k_i] = 0$, $M\left[\Delta k_i^2\right] = \frac{b_i^2}{3}$ ($i = 1, 2, \ldots, s$), from Eqs. (7.41) we get the following relations:

$$a_0 = \left(1 + \frac{5s}{4}\right) Z_0 - \frac{15}{4} \sum_{i=1}^{s} \frac{Z_{ii}^{(2)}}{b_i^2}; \qquad (7.44)$$

$$a_i = \frac{3Z_i^{(1)}}{b_i^2} \quad (i = 1,2, \ldots, s);$$

$$a_{ij} = \begin{cases} \dfrac{9Z_{ij}^{(2)}}{b_i^2 b_j^2} & (i < j = 1,2,\ldots, s), \\ \dfrac{3Z_{ii}^{(2)} - b_i^2 Z_0}{4b_i^4} \cdot 15 & (i = j = 1,2,\ldots, s). \end{cases} \tag{7.44 (cont)}$$

For equal ranges of variation of the elements of vector ΔK, Eqs. (7.43) and (7.44) are considerably simplified and are of the form

$$a_0 = \left(1 + \frac{s}{2}\right) Z_0 - \frac{1}{2\sigma^2} \sum_{i=1}^{s} Z_{ii}^{(2)};$$

$$a_i = \frac{Z_i^{(1)}}{\sigma^2} \quad (i = 1,2,\ldots, s);$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ij}^{(2)}}{\sigma^4} & (i < j = 1,2,\ldots, s), \\ \dfrac{Z_{ii}^{(2)} - \sigma^2 Z_0}{2\sigma^4} & (i = j = 1,2,\ldots, s) \end{cases} \tag{7.45}$$

and

$$a_0 = \left(1 + \frac{5s}{4}\right) Z_0 - \frac{15}{4b^2} \sum_{i=1}^{s} Z_{ii}^{(2)};$$

$$a_i = \frac{3Z_i^{(1)}}{b^2} \quad (i = 1,2,\ldots, s);$$

$$a_{ij} = \begin{cases} \dfrac{9Z_{ij}^{(2)}}{b^4} & (i < j = 1,2,\ldots, s), \\ \dfrac{45Z_{ii}^{(2)} - 15b^2 Z_0}{4b^4} & (i = j = 1,2\ldots, s). \end{cases} \tag{7.46}$$

And, finally, for the normalized elements of vector ΔK ($\sigma = 1$), Eqs. (7.45) and (7.46) become /262

$$a_0 = \left(1 + \frac{s}{2}\right) Z_0 - \frac{1}{2} \sum_{i=1}^{s} Z_{ii}^{(2)}; \tag{7.47}$$

$$a_i = Z_i^{(1)} \quad (i = 1,2,\ldots, s),$$

$$a_{ij} = \begin{cases} \dfrac{Z_{ii}^{(2)} - Z_0}{2} & (i = j = 1,2,\ldots, s), \\ Z_{ij}^{(2)} & (i < j = 1,2,\ldots, s) \end{cases} \tag{7.47 (cont)}$$

and

$$a_0 = \left(1 + \tfrac{5}{4} s\right) Z_0 - \tfrac{15}{4} \sum_{i=1}^{s} Z_{ii}^{(2)};$$

$$a_i = 3Z_i^{(1)} \quad (i = 1,2,\ldots, s);$$

$$a_{ij} = \begin{cases} \dfrac{45Z_{ii}^{(2)} - 15Z_0}{4} & (i = j = 1,2,\ldots, s), \\ 9Z_{ij}^{(2)} & (i < j = 1,2,\ldots, s). \end{cases} \tag{7.48}$$

Eqs. (7.47) and (7.48) are quite simple computationally speaking for computing the coefficients a_0, a_i, and a_{ij} of a second-degree approximating polynomial if we know the numerical values of the quantities Z_0, $Z_i^{(1)}$, $Z_i^{(2)}$. Since computing the exact values of these quantities in accordance with their mathematical expressions (7.42) does not appear possible, then in their place we can use their estimates $\hat{Z}_0$. $\hat{Z}_i^{(1)}$, $\hat{Z}_{ij}^{(2)}$, determined by one of the known approximate methods of statistical analysis. To do this, we must set up the sequence of vectors

$$\Delta K^{(1)}, \Delta K^{(2)}, \ldots, \Delta K^{(N_1)}, \tag{7.49}$$

whose elements satisfy the selected law of the distribution of probability density f (ΔK) (normal or equiprobable), then by computing the function I (ΔK) for each element of sequence (7.49), we can construct the sequence of functions

$$I^{(1)}\left[\Delta K^{(1)}\right], I^{(2)}\left[\Delta K^{(2)}\right], \ldots, I^{(N_1)}\left[\Delta K^{(N_1)}\right]. \tag{7.50}$$

Treating the elements of sequences (7.49) and (7.50) by one of the methods of the statistical analysis of nonlinear systems, we can compute the estimates $\hat{Z}_0$, $\hat{Z}_i^{(1)}$, $\hat{Z}_{ij}^{(2)}$. Thus, when using /263
the method of statistical tests we will have:

$$\left(\hat{Z}_0\right)_{N_1} = \frac{1}{N_1} \sum_{i=1}^{N_1} I^{(i)}\left[\Delta K^{(i)}\right], \tag{7.51}$$

$$\left(\hat{Z}_j^{(1)}\right)_{N_1} = \frac{1}{N_1}\sum_{i=1}^{N_1} I^{(i)}\left[\Delta K^{(i)}\right]\Delta k_j^{(i)},$$

$$\left(\hat{Z}_{jl}^{(2)}\right)_{N_1} = \frac{1}{N_1}\sum_{i=1}^{N_1} I^{(i)}\left[\Delta K^{(i)}\right]\Delta k_j^{(i)}\Delta k_l^{(i)} \quad (j \leqslant j = 1,2,\ldots, s). \tag{7.51 (cont)}$$

Essentially, the only requirement in conducting the above-described computations to arrive at the estimates $(\hat{Z}_o)_{N_1}$, $(\hat{Z}_j^{(1)})_{N_1}$, $(\hat{Z}_l^{(2)})_{N_1}$ is the requirement that the elements of sequence (7.49) satisfy the specified law of the distribution of probability density of vector ΔK.

Analyzing the foregoing, it can be noted that this algorithm of the method of stochastic approximation jointly with Eq. (7.51) for computing estimates leads to the necessity of setting up sequence (7.50). The latter is a quite computationally laborious operation if the function I (ΔK) is the mathematical expectation of the specified random function, for example,

$$I(\Delta K) = M_V[\Phi(V, \Delta K)]. \tag{7.52}$$

To compute each j-th element of sequence (7.50) we obviously must construct the sequence of vectors

$$V^{(1)}, V^{(2)}, \ldots, V^{(N_2)} \tag{7.53}$$

and for each of its elements $V^{(i)}$ we must compute the sequence of functions

$$\Phi^{(1)}\left[\Delta K^{(j)}, V^{(1)}\right],\ \Phi^{(2)}\left[\Delta K^{(j)}, V^{(2)}\right], \ldots, \Phi^{(N_2)}\left[\Delta K^{(j)}, V^{(N_2)}\right]. \tag{7.54}$$

Treatment of the sequence (7.54) for example by the method of statistical tests using the formula

$$\left(\hat{I}\left[\Delta K^{(j)}\right]\right)_{N_2} = \frac{1}{N_2}\sum_{i=1}^{N_2} \Phi^{(i)}\left[\Delta K^{(j)}, V^{(i)}\right] \tag{7.55}$$

then gives the necessary estimate of the required mathematical expectation (7.52).

Overall, to construct a quadratic model of the quality criterion we must perform $\bar{N} = N_1 \times N_2$ integrations of differential equation (7.2) describing the control process. Thus, when $N_1 = N_2 \geqslant 100$, /264 the number $\bar{N} \geqslant 10{,}000$, which means a fairly large volume of computational work and cumbersomeness in this approach to setting up models of the quality criterion or constraints. For a high order s of the vector of the parameters K being optimized, this approach nonetheless can be more applicable than the algorithms described in sections 7.2-7.4. Owing to the necessity of reducing the number $\bar{N}$ of integrations of differential equation (7.2), below we look at several algorithms based on the ratios of the method of stochastic approximation.

To do this, let us introduce the random vector Λ composed of elements of the vector of random perturbations V and elements of the vector ΔK (we assume the randomness of vector ΔK for computational purposes). Then we can set up a polynomial for the function $\Phi(V, \Delta K)$:

$$\begin{aligned}\Phi(V, \Delta K) = a_0 + \sum_{i=1}^{s} a_i \Delta k_i + \sum_{i,j=1}^{s} a_{ij} \Delta k_i \Delta k_j + \\ + \sum_{i=1}^{m} b_i v_i + \sum_{i,j=1}^{m} b_{ij} v_i v_j + \sum_{i=1}^{s} \sum_{j=1}^{m} c_{ij} \Delta k_i v_j + \\ + \sum_{i,j=1}^{m} \sum_{l=1}^{s} c_{ijl} v_i v_j \Delta k_l + \sum_{i,j=1}^{s} \sum_{l,\nu=1}^{m} c_{ijl\nu} \Delta k_i \Delta k_j v_l v_\nu .\end{aligned} \tag{7.56}$$

Performing the operation of averaging polynomial (7.56), we get

$$\begin{aligned}I(\Delta K) = a_0 + \sum_{i=1}^{s} a_i \Delta k_i + \sum_{i,j=1}^{s} a_{ij} \Delta k_i \Delta k_j + \sum_{i=1}^{m} b_i M\left[v_i^2\right] + \\ + \sum_{i=1}^{m} \sum_{l=1}^{s} c_{iil} M\left[v_i^2\right] \Delta k_l + \sum_{i=1}^{m} \sum_{l,\nu=1}^{s} c_{iil\nu} M\left[v_i^2\right] \Delta k_l \Delta k_\nu .\end{aligned} \tag{7.57}$$

To obtain Eq. (7.57), it is assumed that there is no correlation between the elements of the vector Λ. By writing out Eq. (7.57) in the form

$$\begin{aligned}\hat{I}(\Delta K) = a_0 + \sum_{i=1}^{m} b_i M\left[v_i^2\right] + \sum_{l=1}^{s} \left(a_l + \sum_{i=1}^{m} c_{iil} M\left[v_i^2\right] \right) \Delta k_l + \\ + \sum_{i,j=1}^{s} \left(a_{ij} + \sum_{\nu=1}^{m} c_{\nu\nu ij} M\left[v_\nu^2\right] \right) \Delta k_i \Delta k_j ,\end{aligned} \tag{7.58}$$

We can state that we have obtained an approximate quadratic model for quality criterion (7.52). It appears possible to consider a fairly simple algorithm if we propose another method of computing the elements Z_0, $Z_i^{(1)}$, $Z_{ij}^{(2)}$, and so on.

Since

$$Z_0 = M_{\Delta K}[I(\Delta K)],$$

then with reference to Eq. (7.52) for the quality criterion

$$I(\Delta K) = M_V[\Phi(V, \Delta K)]$$

we get

$$Z_0 = M_{\Delta K}[I(\Delta K)] = M_{\Delta K}[M_V[\Phi(V, \Delta K)]] = = M_{\Delta K, V}[\Phi(\Delta K, V)], \tag{7.59}$$

if the elements of vectors V and ΔK are uncorrelated. This assumption is easily satisfied in practice, since randomness is imputed to the vector ΔK formally only for computational purposes.

An analogy, we can describe algorithms also for computing the elements $Z_i^{(1)}$ and $Z_{ij}^{(2)}$. We will have

$$Z_i^{(1)} = M_{\Delta K}[I(\Delta K)\Delta k_i] = M_{\Delta K, V}[\Phi(\Delta K, V)\Delta k_i], \tag{7.60}$$

$$Z_{ij}^{(2)} = M_{\Delta K}[I(\Delta K)\Delta k_i \Delta k_j] = M_{\Delta K, V}[\Phi(V, \Delta K)\Delta k_i \Delta k_j]. \tag{7.61}$$

Essentially, for uncorrelated elements of the vectors V and ΔK, the operation of successive determination of the mathematical expectation of the function $\Phi(V, \Delta K)$, initially by averaging on the set Ω_V of the elements of vector V for fixed values of vector ΔK, and then averaging the results of I (ΔK) on a set $\omega_{\Delta K}$ in accordance with Eqs. (7.59), (7.60), and (7.61), can be replaced with the operation of single averaging of the function $\Phi(V, \Delta K)$ on the set $\Omega_\Delta \subset \Omega_V \cup \omega_{\Delta K}$. In accordance with Eq. (7.59), (7.60), and (7.61), we can write out the formulas for computing the estimates of elements Z_o, $Z_i^{(1)}$, $Z_{ij}^{(2)}$, for example, by the method of statistical tests:

$$\left(\hat{Z}_0\right)_{N_3} = \frac{1}{N_3}\sum_{l=1}^{N_3}\Phi^{(l)}[\Delta K^{(l)}, V^{(l)}],$$

$$\left(\hat{Z}_i^{(1)}\right)_{N_3} = \frac{1}{N_3}\sum_{l=1}^{N_3}\Phi^{(l)}[\Delta K^{(l)}, V^{(l)}]\Delta k_i^{(l)},$$

$$\left(\hat{Z}_{ij}^{(2)}\right)_{N_3} = \frac{1}{N_3}\sum_{l=1}^{N_3}\Phi^{(l)}[\Delta K^{(l)}, V^{(l)}]\Delta k_i^{(l)}\Delta k_j^{(l)} \quad (i \leqslant j = 1, 2, \ldots, s). \tag{7.62}$$

In order to use Eq. (7.62), we must construct two sequences: a sequence of random vectors $[f(\Lambda) = f(V) f(\Delta K)]$

$$\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(N_3)} \tag{7.63}$$

the sequence of numerical values of the functions /266

$$\Phi^{(1)}\left[\Lambda^{(1)}\right], \Phi^{(2)}\left[\Lambda^{(2)}\right], \ldots, \Phi^{(N_3)}\left[\Lambda^{(N_3)}\right], \tag{7.64}$$

computed for the solutions to nonlinear equation (7.2) for each element of sequence (7.63). In accordance with the notation (7.63), we can rewrite Eqs. (7.62) as

$$\begin{gathered}
\left(\hat{Z}_0\right)_{N_3} = \frac{1}{N_3} \sum_{l=1}^{N_3} \Phi^{(l)}\left[\Lambda^{(l)}\right], \\
\left(\hat{Z}_i^{(1)}\right)_{N_3} = \frac{1}{N_3} \sum_{l=1}^{N_3} \Phi^{(l)}\left[\Lambda^{(l)}\right] \lambda_{m+i}, \\
\left(\hat{Z}_{ij}^{(2)}\right)_{N_3} = \frac{1}{N_3} \sum_{l=1}^{N_3} \Phi^{(l)}\left[\Lambda^{(l)}\right] \lambda_{m+i} \lambda_{m+j} \quad (i \leqslant j = 1, 2, \ldots, s).
\end{gathered} \tag{7.65}$$

Let us look at an example of applying Eqs. (7.59) - (7.61) to compute the numerical values Z_o, $Z_i^{(1)}$, $Z_{ij}^{(2)}$, if the function $\Phi(V, \Delta K)$ is specified on the set Ω_Λ with the characteristic σ_V = const = 1, $\sigma_{\Delta K}$ = const = 1, and is of the form

$$\begin{gathered}
\Phi(V, \Delta K) = 0{,}5 + 0{,}1 v_1^2 \Delta k_1^2 + 0{,}2 v_1^4 \Delta k_1 v_2^2 + 0{,}4 v_1^2 v_2^2 \Delta k_2 + \\
+ 0{,}3 v_1^4 v_2^4 \Delta k_2^2 + 0{,}05 v_1^2 v_2^6 \Delta k_1 \Delta k_2 + 0{,}01 v_1^4 v_2^2 \Delta k_2^2 \Delta k_1^2.
\end{gathered}$$

Computing I (ΔK) in accordance with Eq. (7.52), we get

$$\begin{gathered}
I(\Delta K) = M_V\left[\Phi(\Delta K, V)\right] = 0{,}5 + 0{,}1 \Delta k_1^2 + 0{,}6 \Delta k_1 + 0{,}4 \Delta k_2 + \\
+ 2{,}7 \Delta k_2^2 + 0{,}75 \Delta k_1 \Delta k_2 + 0{,}03 \Delta k_1^2 \Delta k_2^2.
\end{gathered} \tag{7.66}$$

Using this result (7.66), let us compute in accordance with Eq. (7.62) the quantities necessary in constructing the quadratic model:

$$\begin{gathered}
Z_0 = 3{,}33;\ Z_1^{(1)} = 0{,}6;\ Z_2^{(1)} = 0{,}4;\ Z_{11}^{(2)} = 3{,}59; \\
Z_{12}^{(2)} = 0{,}75;\ Z_{22}^{(2)} = 8{,}89.
\end{gathered} \tag{7.67}$$

Obviously, the numerical value (7.67) can be obtained by using Eqs. (7.59) - (7.61). Thus, we will have

$$\begin{gathered} Z_0 = 3{,}33;\ Z_1^{(1)} = 0{,}6;\ Z_2^{(1)} = 0{,}4; \\ Z_{11}^{(2)} = 3{,}59;\ Z_{12}^{(2)} = 0{,}75;\ Z_{22}^{(2)} = 8{,}89. \end{gathered} \tag{7.68}$$

The numerical values of the elements Z_o, $Z_i^{(1)}$, $Z_{ij}^{(2)}$ in accordance with the formulas (7.67) and (7.68) we have obtained coincide, however in the case of (7.68) the calculations proved to be simpler by virtue of the use of the single operation of mathematical expectation.

Let us compute the coefficients a_o, a_i, and a_{ij} in accordance with the model (7.47). We will have /267

$$\begin{gathered} a_0 = 0{,}42;\quad a_1 = 0{,}6;\quad a_2 = 0{,}4; \\ a_{11} = 0{,}13;\quad a_{12} = 0{,}75;\quad a_{22} = 1{,}78. \end{gathered}$$

In accordance with the foregoing, the quadratic model of the quality criterion can be represented by the equation

$$\begin{gathered} I(\Delta K) = 0{,}42 + 0{,}6\Delta k_1 + 0{,}4\Delta k_2 + 0{,}75\Delta k_1\Delta k_2 + \\ + 0{,}13\Delta k_1^2 + 1{,}78\Delta k_2^2. \end{gathered} \tag{7.69}$$

A similar quadratic model of the quality criterion can be constructed also for the equiprobable distribution of the probability density of the elements of vector ΔK. Thus, when b = 1, we get:

$$\begin{gathered} Z_0 = 1{,}436;\ Z_1^{(1)} = 0{,}2;\ Z_2^{(1)} = 0{,}133; \\ Z_{11}^{(2)} = 0{,}489;\ Z_{12}^{(2)} = 0{,}083;\ Z_{22}^{(2)} = 0{,}72, \end{gathered}$$

and, this means, that when

$$\begin{gathered} a_0 = 4{,}53;\quad a_1 = 0{,}6;\quad a_2 = 0{,}4; \\ a_{11} = 0{,}11;\quad a_{12} = 0{,}75;\quad a_{22} = 2{,}71 \end{gathered}$$

the model of the quality criterion is of the form

$$\begin{gathered} \hat{I}(\Delta K) = 4{,}53 + 0{,}6\Delta k_1 + 0{,}4\Delta k_2 + 0{,}75\Delta k_1\Delta k_2 + \\ + 0{,}11\Delta k_1^2 + 2{,}71\Delta k_2^2. \end{gathered} \tag{7.70}$$

Models of quality criteria (7.69) and (7.70) differ from each other, as do the results of searching for an extremum for these models

$$\Delta\tilde{k}_1^{(H)} = -4{,}9;\ \Delta k_2^{(H)} \approx 0{,}903;$$
$$\Delta\tilde{k}_1^{(p)} = -3{,}75;\ \Delta\tilde{k}_2^{(p)} \approx 0{,}304.$$

Therefore selection of the law of the probability distribution of vector ΔK and the statistical characteristics of its element constitutes one of the principal problems in this algorithm based on the method of stochastic approximation.

Thus, essentially this method boils down to expanding the space of random vectors Ω_V of this process by imputing random properties, for computational purposes, to the vector of increments ΔK of the parameters K being optimized and setting up the random vector ΔK. Treatment of the results of the output coordinates of the controlled process (7.2) obtained when it is acted on by the realizations of the enlarged vector Λ enables /268
us to carry out the above-indicated computations and to construct the model of the quality criterion and the constraints necessary for optimization.

An advantage of this method for setting up models of the quality criteria and constraints on the set of parameters being optimized lies in the considerable reduction of the volume of computation through the single deriving and treating of the random sequences (7.63) and (7.64) instead of the sequences (7.53), (7.54), (7.49), and (7.50).

Returning to the quadratic method of the quality criterion

$$\hat{I}(\Delta K) = A + B^{*}\Delta K + \Delta K^{*}C\Delta K, \tag{7.71}$$

where

$$A = a_0,\ B = (a_i),\ C = (a_{ij}),$$

written in matrical form, we should note that by virtue of employing the approximate methods of statistical analysis for computing the coefficients of the approximating polynomial, the elements of vector B and matrix C contain random components ΔB and ΔC. Therefore Eq. (7.71) can be written as

$$\hat{I} = A + \left(\tilde{B} + \Delta B\right)^{*}\Delta K + \Delta K^{*}\left(\tilde{C} + \Delta C\right)\Delta K. \tag{7.72}$$

The condition of the extremum of quadratic form (7.72) here is of the form

$$\Delta\tilde{K} = -\frac{1}{2}\left(\tilde{C}+\Delta C\right)^{-1}\left(\tilde{B}+\Delta B\right). \tag{7.73}$$

The random component of the matrix C can to a considerable extent determine the fact of the existence of a matrix that is the reciprocal of matrix C, and this means the soundness of the expression (7.73). Since in computing the elements of matrices B and C we can estimate the possible maximum value of the error $|\Delta c_{ij}| \leqslant \varepsilon_{ij}$ and $|\Delta b_i| \leqslant \varepsilon_i$, it appears possible when necessary to organize a correction to matrix C such that by changing its elements c_{ij} by the quality $|\Delta \bar{c}_{ij}| \leqslant \varepsilon_{ij}$, we can ensure that the conditions for the existence of the reciprocal matrix C are met. Denoting the corrected matrix by $\bar{C}$, we can write conditions (7.73) in the form

$$\Delta\tilde{K} = -\frac{1}{2}\bar{C}^{-1}\tilde{B} - \frac{1}{2}\bar{C}^{-1}\Delta B. \tag{7.74}$$

The second term in Eq. (7.74) then will characterize the error $\delta\Delta K = \bar{C}^{-1}\Delta B$ in the computing of vector $\Delta\tilde{K}$ at each step of optimization. The foregoing indicates the necessity of the statistical analysis of the results of optimation obtained by employing numerical iterative methods.

In conclusion, let us look at the problem of constructing quadratic models for quality criteria of the form /269

$$I_1(\Delta K) = M_V\left[\Phi^p(V, \Delta K)\right], \tag{7.75}$$

$$I_2(\Delta K) = M_V\left[\int_0^T \Phi^p(V, \Delta K)\,dt\right], \tag{7.76}$$

$$I_3 = M_V\left[\{\Phi(V, \Delta K) - M_V[\Phi(V, \Delta K)]\}^2\right], \tag{7.77}$$

quite often encountered in problems of the statistical optimization of control processes.

We can write expressions analogous to Eqs. (7.59) - (7.61) for the quality criterion (7.75), when $p \geqslant 2$. We will have:

$$\begin{gathered} Z_0 = M_{\Delta K, V}\left[\Phi^p(V, \Delta K)\right], \\ Z_i^{(1)} = M_{\Delta K, V}\left[\Phi^p(V, \Delta K)\,\Delta k_i\right], \\ Z_{ij}^{(2)} = M_{\Delta K, V}\left[\Phi^p(V, \Delta K)\,\Delta k_i \Delta k_j\right]. \end{gathered} \tag{7.78}$$

We can proceed in like manner also in computing the vector $Z = \{Z_0, Z_i^{(1)}, Z_{ij}^{(2)}\}$ for the quality criterion (7.76). Here, we get:

$$Z_0 = M_{V,\Delta K}\left[\int_0^T \Phi^p(V, \Delta K, t)\,dt\right],$$

$$Z_i^{(1)} = M_{V,\Delta K}\left[\int_0^T \Phi^p(V, \Delta K, t)\,dt\Delta k_i\right],$$

$$Z_{ij}^{(2)} = M_{V,\Delta K}\left[\int_0^T \Phi^p(V, \Delta K, t)\,dt\Delta k_i\Delta k_j\right]. \tag{7.79}$$

The situation is somewhat more complicated when we are dealing with the quality criterion (7.77). Transforming Eq. (7.77), we get

$$I_3 = M_V[\Phi^2(V, \Delta K)] - (M_V[\Phi(V, \Delta K)])^2. \tag{7.80}$$

Since our developed apparatus cannot be applied directly to Eq. (7.80) by virtue of the nonlinearity of the above-described transformation, in this case we can construct quadratic models for the criteria

$$M_V[\Phi^2(V, \Delta K)] \text{ and } M_V[\Phi(V, \Delta K)]$$

in the form

$$\tilde{M}_V[\Phi^2(V, \Delta K)] = a_0 + (a^{(1)})^*\Delta K + \Delta K^* a^{(2)}\Delta K,$$

$$\tilde{M}_V[\Phi(V, \Delta K)] = b_0 + (b^{(1)})^*\Delta K + \Delta K^* b^{(2)}\Delta K. \tag{7.81}$$

Substituting Eqs. (7.81) into (7.80), we can get an approximate formula for the quality criterion (7.77) in the form /270

$$\tilde{I}_3 \approx (a_0 - b_0^2) + [a^{(1)} + 2b_0 b^{(1)}]^*\Delta K + \\ + \Delta K^*[a^{(2)} - 2b_0 b^{(2)} - b^{(1)}(b^{(1)})^*]\Delta K. \tag{7.82}$$

We can similarly carry out computations for the integral criteria of the form

$$I = M_V\left[\int_0^T [\Phi(V, \Delta K) - M_V[\Phi(V, \Delta K)]]^2 dt\right].$$

7.6. Methods of Allowing for Constraints in Problems of Optimizing Control Systems

Above we formulated the problem 7.2 of the multidimensional optimization of an automatic system described by the nonlinear stochastic differential equations. The methods of computing the numerical values of the quality criterion and the constraints, which are statistical characteristics of the assigned functions of solutions to the stochastic nonlinear differential equations, were discussed in Chapter Five. Below we examine the problem of nonlinear programming, assuming that all the necessary computations can be successfully carried out by using the above-presented methods, in the form of problem 7.2. In examining the methods of the solution of problem 7.2 we will assume that the vector $\tilde{K}$ sought for does exist. Problem 7.2 boils down to general problems of nonlinear programming, for which there are as yet no common methods and solution algorithms if nothing is known in advance about the nature of the functions I, Q_1, Q_2, ..., Q_ℓ.

The quality criterion I(K) in this problem can have several extrema, and the constraints can be nonconvex functions of parameters of the control K. This feature of the problem led to the development of a long series of approximate methods and algorithms for solving problems in nonlinear programming employing both the determinate as well as the random search for the optimal solution /61, 37/.

Let us describe several particular problems in nonlinear programming that can be posited as the basis of the approximate methods of solving the nonlinear programming problem of the form 7.2.

Problem of nonlinear programming. If the quality criterion (7.1) and the constraints Q_1, Q_2, ..., Q_1 are linear forms of the sought-for parameters:

$$I = a_0 + \sum_{i=1}^{s} a_i k_i, \qquad (7.83)$$

$$Q_j = b_{j,0} + \sum_{i=1}^{s} b_{j,i} k_i \quad (j = 1, 2, \ldots, l), \qquad (7.84)$$

then the problem of determining the optimal vector $\tilde{K}$ is a linear programming problem, which can be mathematically formulated thusly: /271

Problem 7.3. Find the optimal solution $\tilde{K}$ from the condition that the quality criterion (7.83) is a minimum given the constraint (7.84) and the constraints of the form (7.5).

The methods of solving linear programming problems are quite well elaborated. The principal method of solving the problems in linear programming is the simplex method.

Problem of quadratic programming. If the quality criterion (7.1) is quadratic in form, as follows

$$I(K) = a_0 + \sum_{i=1}^{s} a_i k_i + \sum_{i,j=1}^{s} a_{ij} k_i k_j, \tag{7.85}$$

and if the constraints are linear relations of the form (7.84) and (7.5), then the problem of determining the optimal solution $\tilde{K}$ is a problem in quadratic programming. Essentially it amounts to the following.

Problem 7.4. Find the optimal solution K from the condition that the quality criterion (7.85) is a minimum, given the constraints (7.5) and (7.84). The methods of solving problems in quadratic programming are adequately elaborated, however known methods in algorithms require convexities or rigorous convexity of the quadratic quality criterion (7.85) being optimized.

Searching for an extremum with equality constraints. If the constraints (7.6) are the quality constraints of the form

$$Q_i = Q_i^0 \quad (i = 1, 2, \ldots, l), \tag{7.86}$$

then the problem of searching for the optimal solution $\tilde{K}$ is solved by employing Lagrange multipliers, as the problem in searching for the extremum of the function

$$H = \min_K \left\{ I(K) + \sum_{i=1}^{l} \lambda_i [Q_i - Q_i^0] \right\}. \tag{7.87}$$

Since the Lagrange multipliers are not specified in advance, here to determine them we must then use the relations (7.86). In the case of the quadratic quality criterion (7.85) and linear equality constraints, this problem is solved by employing the following algorithm. By writing out Eqs. (7.85) and (7.84) in matrical form, we will have

$$I = a_0 + a^* K + K^* A K, \tag{7.88}$$

$$Q = b_0 + BK, \tag{7.89}$$

where /272

$$a_0 \text{ is a number; } a^* = (a_1 a_2 a_3 \ldots a_s);$$
$$A = (a_{ij});\ B = (b_{ij});$$
$$b_0^* = (b_{01} b_{02} \ldots b_{0s}).$$

Since in this case Eq. (7.87) is defined by the formula

$$H = \{a_0 + a^* K + K^* A K + \Lambda^* (b_0 + BK)\},$$
$$\Lambda^* = (\lambda_1 \lambda_2 \ldots \lambda_s),$$

the conditions of the optimality of quality criterion (7.88) are of the form

$$\left(\frac{\partial H}{\partial K}\right)^* = a + 2AK + B^* \Lambda = 0,$$

whence

$$K = -\frac{1}{2} A^{-1} (a + B^* \Lambda). \tag{7.90}$$

By substituting Eq. (7.90) into Eq. (7.89), let us write out an equation for determining the vector of Lagrange multipliers

$$b_0 - \frac{1}{2} B A^{-1} (a + B^* \Lambda) = Q_0,$$

whence we can obtain the following equality:

$$\Lambda = 2 (B A^{-1} B^*)^{-1} \left[b_0 - Q_0 - \frac{1}{2} B A^{-1} a \right]. \tag{7.91}$$

And, finally, by substituting Eq. (7.91) into (7.90), we get a working formula for computing the optimal coefficients

$$K = -\frac{1}{2} A^{-1} a - A^{-1} B^* (B A^{-1} B^*)^{-1} \left[b_0 - Q_0 - \frac{1}{2} B A^{-1} a \right]. \tag{7.92}$$

we are seeking.

In this example, Eqs. (7.90), (7.91), and (7.92) are valid if matrix A is nonsingular. If here the constraints are described by second-order equations of the form

$$g = b_0 + b^*K + K^*BK = g_0 \quad (7.93)$$

(g is a scalar), the problem of searching for the optimal solution $\tilde{K}$ is already considerably complicated, since in this case we will have

$$K = -\frac{1}{2}(A + \Lambda B)^{-1}(a + \Lambda B); \quad (7.94)$$

/273

$$b_0 + b^*\left\{-\frac{1}{2}(A + \Lambda B)^{-1}(a + \Lambda B)\right\} + \\ + \left\{-\frac{1}{2}(A + \Lambda B)^{-1}(a + \Lambda B)\right\}^* B\left\{-\frac{1}{2}(A + \\ + \Lambda B)^{-1}(a + \Lambda B)\right\} = g_0. \quad (7.95)$$

Eq. (7.94) and (7.95) are nonlinear equations in the numerical value of the Lagrange multiplier Λ. To solve Eq. (7.95), it is now necessary to use numerical methods. When ℓ equality constraints exist, we will have a system of fourth-degree nonlinear equations.

There are several algorithms based on reducing the problem of nonlinear programming with inequality constraints to a problem in searching for an extremum of one function of many variables using "penalty" functions. Let us introduce a miscoordination vector in meeting the constraints

$$\delta Q = Q(K) - Q_0$$

and "penalty" functions

$$\varphi(\delta Q_i) \; i = 1, 2, \ldots, l.$$

Then we can seek the solution to problem 7.2 as a solution to a problem of searching for the minimum of the function

$$H = \min_K \left\{ I(K) + \sum_{i=1}^{n} \varphi_i(\delta Q_i)\, \delta Q_i(K) \right\},$$

$$H = \min_K \left\{ I(K) + \sum_{i=1}^{l} \varphi_i(\delta Q_i) \right\}. \quad (7.96)$$

The "penalty" functions can be of the form

$$\varphi_i(\delta Q_i) = \begin{cases} h_i & \delta Q_i > 0, \\ 0 & \delta Q_i \leqslant 0; \end{cases}$$

$$\varphi_i(\delta Q_i) = \begin{cases} (\delta Q_i)^{2h_i} & \delta Q_i > 0, \\ 0 & \delta Q_i \leqslant 0, \end{cases}$$

$$\varphi_i(\delta Q_i) = e^{h_i \delta Q_i}.$$

and so on.

Here h_i are fairly large positive numbers.

The problem of searching for an extremum of Eq. (7.96) can be solved by familiar methods of searching for an extremum of functions of many variables.

Of interest is an approach to solving problem 7.2 using algorithms of the successive refinement of solutions based on algorithms from linear and quadratic programming. The method of successive optimization for solving problems in nonlinear programming is given in a study by V. M. Ponomarev /59/. Essentially, the method of successive optimization amounts to the initial problem in nonlinear programming being replaced with a sequence of problems in quadratic programming which was examined above. Since the method of successive optimization has found wide use in solving problems of the parametric synthesis of control systems of flight vehicle motion, control systems for industrial processes, and so on, and is fairly well known to a wide range of specialists in the field of control theory, we omit presenting the scientific and methodological fundamentals and the computational features of the method of successive optimization within the framework of this present book. /274

We can similarly construct a process of successive optimization based on algorithms in linear programming. Let us explain the foregoing by assuming that the quality criterion and the constraints are convex or rigorously convex functions.

Suppose we have selected an initial (figurate) point K_0^0 in the parameter space Ω_K. In the neighborhood of this point we can write out expansions of the form

$$I = I(K_0^0) + \left(\frac{\partial I}{\partial K}\right)^*_{K_0^0} \Delta K + \frac{1}{2} \Delta K^* \left(\frac{\partial^2 I}{\partial K^2}\right)_{K_0^0} \Delta K + \ldots \quad (7.97)$$

$$Q_i = Q_i(K_0^0) + \left(\frac{\partial Q_i}{\partial K}\right)^*_{K_0^0} \Delta K + \frac{1}{2} \Delta K^* \left(\frac{\partial^2 Q_i}{\partial K^2}\right)_{K_0^0} \Delta K + \ldots \quad (7.98)$$

for the quality criterion and the constraints, where ΔK is the s-dimensional vector-column of the increments of parameters $\Delta K = K - K_0^0$, $I(K_0^0)$, $Q_i(K_0^0)$, $i = 1, 2, \ldots, s$ are the values of the quality criterion and the constraints computed at the points K_0^0, $\frac{\partial I}{\partial K}, \frac{\partial Q_i}{\partial K}$, $\frac{\partial^2 I}{\partial K^2}, \frac{\partial^2 Q_i}{\partial K^2}$ — which are, respectively, the vector row and the matrix of the projections of the first- and second-order gradients of the quality criterion and the constraints, and so on.

Limiting ourselves only to the linear terms of expansion (7.97) and (7.98), we can formulate the following problem in linear programming:

$$I = \min_{\Delta K} \left\{ I(K_0^0) + \left[\frac{\partial I}{\partial K}(K_0^0) \right] \Delta K \right\},$$

$$Q_i(K_0^0) + \left[\frac{\partial Q_i}{\partial K}(K_0^0) \right] \Delta K \leqslant Q_i^0 \quad (i = 1, 2, \ldots, l). \tag{7.99}$$

Constraints (7.5) can be represented in the form

$$\check{K} \leqslant K_0^0 + \Delta K \leqslant \hat{K}. \tag{7.100}$$

Naturally, expansions (7.99) are valid in some closed parameter space /275

$$(\Delta K) \leqslant W, \tag{7.101}$$

where W is the vector column of constant numbers.

In this problem the vector W is unknown and its determination requires considerable computations.

Assuming that expressions (7.100) can be determined, let us write out the process of searching for vector $\tilde{K}$ supplying a minimum to the quality criterion (7.1) for the constraints (7.5) and (7.6).

For the sequence

$$W_1^0 < W_2^0 < \ldots < W_l^0 < \ldots \tag{7.102}$$

we find the sequence of vectors

$$K_0^0, K_1^0 = K_0^0 + \Delta K_1^0, \ldots, K_l^0 = K_{l-1}^0 + \Delta K_l^0, \ldots \tag{7.103}$$

and, this means, the sequence of numerical values of the quality criterion

$$I(K_0^0) > I(K_1^0) > \ldots > I(K_i^0) > \ldots \tag{7.104}$$

and the sequence of numerical values of the constraints

$$\begin{gathered} Q_p(K_0^0) \leqslant Q_p^0,\ Q_p(K_1^0) \leqslant Q_p^0, \ldots, Q_p(K_i^0) \leqslant Q_p^0, \ldots \\ (p = 1, 2, \ldots, l) \end{gathered} \tag{7.105}$$

by solving the sequence of problems in linear programming

$$\begin{gathered} I = \min_{\Delta K} \left\{ I(K_j^0) + \left[\frac{\partial I}{\partial K}(K_0^0)\right]^* \Delta K \right\}, \\ Q_p(K_j^0) + \left[\frac{\partial Q_p}{\partial K}(K_0^0)\right]^* \Delta K \leqslant Q_p^0 \quad (p = 1, 2, \ldots, l), \\ \check{K} \leqslant K_{j-1}^0 + \Delta K \leqslant \hat{K}, \\ (\Delta K) \leqslant W_j^0 \quad (j = 1, 2, \ldots). \end{gathered} \tag{7.106}$$

The computations of the sequences (7.104) and (7.105) end at the i-th element of sequence (7.102) when one of the conditions of sequences (7.104) and (7.105) is violated.

Let us call the constructing of the i-th sequence (7.102) a stage in the solution of the optimization problem, and the solution of the linear programming problem at the i-th stage for the j-th element -- a step.

Since the values of the quality criterion and the constraints are computed at the end of each optimization step for the construction of sequences (7.104) and (7.105), these values are used at the new optimization step for revising the linear programming problem (7.106). Gradients of the quality criterion and of the constraints are computed only at the first step of each optimization stage. /276

At the point

$$K_0^1 = K_{i-1}^0 + \Delta K_i^0$$

the linear programming problem is formed again, that is, its elements $\frac{\partial I}{\partial \Delta K}(K_0^1), \frac{\partial Q_p}{\partial \Delta K}(K_0^1)$ are computed.

For the new sequence

$$w_0^1 < w_1^1 < w_2^1 < \ldots < w_j^1 < \ldots \tag{7.107}$$

and

$$K_0^1;\ K_1^1=K_0^1+\Delta K_1^1;\ K_2^1=K_0^1+\Delta K_2^1;\ \ldots;\ K_j^1=K_0^1+\Delta K_j^1;\ \ldots$$

a solution is carried out for the linear programming problems with $W = w_i^1$ (i = 1, 2, ...):

$$\begin{gathered} I=\min_{\Delta K}\left\{I(K_i^1)+\left[\frac{\partial I}{\partial \Delta K}(K_0^1)\right]^*\Delta K\right\}; \\ Q_p=Q_p(K_i^1)+\left[\frac{\partial Q_p}{\partial \Delta K}(K_0^1)\right]^*\Delta K\leqslant Q_{0p},\ p=1,2,\ldots,l; \\ \check{K}\leqslant K_{j-1}^0+\Delta K_i^1\leqslant \hat{K}; \\ |\Delta K_i|\leqslant w_i^1\quad (i=1,2,\ldots) \end{gathered} \tag{7.108}$$

and at the points of the parameter space

$$K_0^1;\ K_1^1;\ K_2^1;\ \ldots;\ K_j^1;\ \ldots,$$

corresponding to the solutions obtained, the values of functional I(K) and constraints $Q_p(K)$ are computed based on exact formulas, and the following sequences are constructed:

$$\begin{gathered} I_0^1(K_0^1)>I_1^1(K_1^1)>\ldots>I_j^1(K_j^1)>\ldots, \\ Q_{p0}^1(K_0^1)\leqslant Q_{0p}; \\ Q_{p1}^1(K_1^1)\leqslant Q_{0p};\ \ldots;\ Q_{pj}^1(K_j^1)\leqslant Q_{0p};\ \ldots \end{gathered} \tag{7.109}$$

Violation of one of the conditions (7.109) at the j-th element of sequence (7.107) serves as a signal for concluding the process of solving the linear programming problems (7.108) and the formation of a new linear programming problem at the point $K_0^2=K_0^1+\Delta K_j^1$.

Thus, a sequence of numerical values of the quality criterion

$$I(K_0^0)>I(K_0^1)>\ldots>I(K_0^\nu)>\ldots \tag{7.110}$$

is constructed.

If the elements of sequence (7.110) standing one after the other differ by a prespecified quantity ϵ, we can halt the solution process and assume that the vector of K_0^ν converges at the ϵ- /277
neighborhood of $\tilde{K}$, providing a minimum to the quality criterion (7.1) given the constraint (7.5) and (7.6).

For a complete description of the procedure we must dwell on the problem of forming the sequence W = $\{w_j^i\}$.

To solve the question of selecting sequence

$$w_1^i < w_2^i < \ldots < w_j^i < \ldots \tag{7.111}$$

we can make use, for example, of analogs of gradient methods.

If each i-th sequence (7.111) consists of a single step, then the gradient method can be reduced to this process of optimization, as an analog. Here, at each step we must determine the gradients of the quality criterion and the constraints in the linear programming problem. Suppose the magnitude of the step is determined by the formula

$$\rho^2 - \sum_{i=1}^{s} \Delta k_i^2 = 0. \tag{7.112}$$

If the magnitude of the step ρ is selected, condition (7.112) can be used in forming the vector W.

For example, w_{jp}^{i} can be determined by the formula

$$w_{jp}^i = \frac{\rho_j \frac{\partial I}{\partial K_p}(K_0^i)}{\left[\sum_{\nu=1}^{s}\left\{\frac{\partial I}{\partial k_\nu}(K_0^i)\right\}^2\right]^{\frac{1}{2}}} \quad (p = 1, 2, \ldots, s). \tag{7.113}$$

Eq. (7.113) defines the vector W at each optimization stage and can be used if at this stage constraints (7.6) are satisfied. If the i-th stage consists of μ stages and if conditions (7.6) are satisfied, the problem of constructing sequence (7.111) can proceed on analogy with the method of steepest descent. In several cases the constraint (7.6) can be divided into two groups

$$Q_l - Q_l^0 = 0 \quad (l = 1, 2, \ldots, l_1), \tag{7.114}$$

$$Q_l < Q_l^0 < 0 \quad (l = b_1 + 1, \ldots, l). \tag{7.115}$$

Then the elements of vector W are defined by the formulas

$$w_{jp}^i = \frac{\left[\frac{\partial I}{\partial k_p}(K_0^i) + \sum_{\nu=1}^{l_1} \eta_\nu \frac{\partial Q_\nu}{\partial k_p}(K_0^i)\right]\rho_i}{\left[\sum_{\nu=1}^{s}\left\{\frac{\partial I}{\partial k_\nu}(K_0^i) + \sum_{\nu=1}^{l_1} \eta_1 \frac{\partial Q_1}{\partial k_\nu}(K_0^i)\right\}^2\right]^{\frac{1}{2}}}. \tag{7.116}$$

Lagrange multipliers η_ν are the solution to the system of algebraic equations /278

$$\alpha_r^l + \sum_{\nu=1}^{l_1} \beta_{r\nu}^l \eta_\nu = 0 \quad (r = 1, 2, \ldots, l_1),$$

where

$$\alpha_p^l = \sum_{\nu=1}^{s} \frac{\partial Q_p}{\partial k_\nu}(K_0^l) \frac{\partial I}{\partial k_\nu}(K_0^l),$$

$$\beta_{pj}^l = \sum_{\nu=1}^{s} \frac{\partial Q_p}{\partial k_\nu}(K_0^l) \frac{\partial Q_j}{\partial k_\nu}(K_0^l).$$

Here, constraints (7.115) and (7.100) will be taken into account automatically when solving linear programming problems.

This successive procedure utilizing the formalized apparatus of linear programming is quite effective when digital computers are used. Its main disadvantage is that in it the sequence of numerical values of vector W is not strictly defined. Its advantage is the relative simplicity of the linear representation of the criterion and the constraints.

The V. M. Ponomarev method of successive optimization presupposes constructing a sequence of solutions in the parameter space Ω_K

$$K_0^0,\ K_0^1,\ \ldots,\ K_0^l,\ \ldots$$

by solving the sequence of problems in quadratic programming, of the form

$$I = \min_{\Delta K}\left\{I(K_0^l) + \left[\frac{\partial I}{\partial K}(K_0^l)\right]^* \Delta K + \frac{1}{2}\Delta K^* \frac{\partial^2 I}{\partial K^2}(K_0^l)\Delta K\right\},$$

$$Q(K_0^l) + \left[\frac{\partial Q}{\partial K}(K_0^l)\right]^* \Delta K \leqslant Q^0,$$

$$\check{K} \leqslant K_0^l + \Delta K \leqslant \hat{K}.$$

Formulating problems in quadratic programming is much more complicated than forming problems in linear programming. However, the convergence of the process of searching for solutions to problem 7.2 in this case is not determined by subjective factors, as in the sequence of linear programming problems. By employing methods of constructing models of the quality criterion and constraints examined above, we can successfully apply the algorithms described in this section to optimizing control systems of flight vehicles moving in the earth's atmosphere.

CHAPTER EIGHT

/279

STATISTICAL PREDICTION PROBLEMS OF CONTROLLING THE MOTION OF FLIGHT VEHICLES IN DENSE ATMOSPHERIC LAYERS

8.1. Control of Motion of Flight Vehicles in the Atmosphere with the Prediction of Phase Coordinates

In a large number of cases the control of flight vehicle motion in the dense atmospheric layers described by a system of nonlinear stochastic equations

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_m),$$
$$x_i(t_0) = x_{i,0} \quad (i = 1, 2, \ldots, n) \tag{8.1}$$

is constructed so that the vector of reference control U(t) is specified, and the correction of the phase coordinates of the process is used at specific instants of time $t_1, t_2, \ldots, t_p \in [\bar{t}_0, T]$ by applying to the system (8.1) the controls $\Delta U = U - U$ acting along the time interval ΔT. For definiteness, we will assume that the duration of the correcting controls U is constant and equal to the magnitude of ΔT.

The value of the correcting controls is usually bounded

$$|\Delta U(t_i)| \leqslant \Delta U_i^0 \quad (i = 1, 2, \ldots, p), \tag{8.2}$$

where ΔU_i^o are specified constants.

Suppose the correcting control is selected on the condition that a minimum value is provided for the quality criterion

$$I = \Phi(\Delta X), \tag{8.3}$$

characterizing the scatter of the process either at the next instant of correction $t = t_{i+1}$ or at the time instant when the control of the flight vehicle is terminated.

Suppose for a specified reference control we know the phase state of process (8.1) for the unperturbed motion $X(t) = \tilde{X}(t)$. The effect of the perturbing factors leads to the deviation of the process phase coordinates from the reference values characterized by the mismatch vector

$$\Delta X(t) = X(t) - \tilde{X}(t).$$

The vector ΔX is a function of the random perturbing factors (vector V). Owing to the randomness of vector $\Delta X(t)$, the correcting impulses $\Delta U(t_i)$, $(i = 1, 2, \ldots, p)$ must be functions of the elements of the mismatch vector. /280

The solution to the problem of constructing the optimal control of process (8.1) can be approached in several ways.

One involves determining the function

$$\Delta U(t_i) = \varphi[\Delta X(t_i)] \quad (i = 1, 2, \ldots, p). \tag{8.4}$$

In this case, corresponding to the state of the process $\Delta X(t)$ at the instants of correction t_i $(i = 1, 2, \ldots, p)$, Eq. (8.4) is brought into correspondence to the value of the correcting control. However, this control algorithm (8.4) can prove to be inadequately farsighted.

The second approach to forming the control actions consists of using the estimate of the mismatch vector of the phase coordinates $\Delta\hat{X}(t)$ at the instant of time corresponding to the next instant of correction $t = t_{i+1}$. Then the control algorithm becomes:

$$\Delta U(t_i) = \varphi\left[\Delta\hat{X}(t_{i+1}), t_i\right] \quad (i = 1, 2, \ldots, p). \tag{8.5}$$

And, finally, the control algorithm can be constructed by using the estimate of the mismatch vector $\Delta\hat{X}(t)$ at the instant of termination of the process, $t = T$. We will have

$$\Delta U(t_i) = \varphi\left[\Delta\hat{X}(T), t_i\right] \quad (i = 1, 2, \ldots, p). \tag{8.6}$$

Thus, problems of optimal control of the process (8.1) examined above can be formulated as follows.

Problem 8.1. For the process described by the system of nonlinear stochastic equations (8.1) we must find the structure and the parameters of control (8.4) providing a minimum value for the quality criterion of the process (8.3).

Problem 8.2. It is necessary, based on data of instantaneous measurements of vector $\Delta X(t)$ over the time interval between the correcting impulses to predict the vector at the next instant of correction $t = t_{i+1}$ for the process described by the system of nonlinear stochastic equations (8.1), and to find the structure and parameters of the control (8.5) providing a minimum value for the quality criterion of process (8.3).

Problem 8.3. It is necessary to make the prediction of the terminal state of vector $\Delta X(T)$, for the process described by the system of equations (8.1), based on the data of measurements of vector $\Delta X(t)$ over the interval between the correcting impulses, and to find the structure and parameters of control (8.6) on the condition that the numerical value of the quality criterion (8.3) is minimized.

To solve problems 8.2 and 8.3, we must examine the algorithms /281
for predicting the future phase states of the process based on instantaneous measurement data. Let us examine the solution of the prediction problem.

8.2. Predicting Phase Coordinates of Nonlinear Stochastic Processes

This problem in predicting the future states of the motion of flight vehicles in the dense atmospheric layers based on instantaneous measurements data is one of the problems of making estimates of random processes.

A fairly large number of studies /2, 18, 27, 48, 83, 100/ have dealt with estimates of random processes. One of the most appreciable results in this field was obtained by Weiner who set forth the solution to the problem of filtration and outpacing for the problem of stationary processes with optimal spectra. The work of Weiner was followed by numerous generalizations (for example, /5/) in which the problem of obtaining an optimal linear stationary or nonstationary dynamic system for carrying out smoothing, filtration, or outpacing of the stationary or nonstationary random processes with finite or infinite observation time was examined. In these studies the optimal system is described by the integral Weiner-Hopf equation.

In papers by Kalman /27, 83/ differential equations were obtained for an optimal dynamic system. These results are related with dynamic models of processes of filtration and outpacing of random processes.

Let us examine algorithms for predicting the future states of stochastic processes described by nonlinear equations (8.1)

Let us assume that in the interval of observation $t \in [t_1, t_2]$ of process (8.1), several functions

$$\eta_{l1}(\Delta X, t), \quad \eta_{l2}(\Delta X, t), \quad \ldots, \quad \eta_{ll}(\Delta X, t). \tag{8.7}$$

are measured.

Owing to the dependence of the solutions to Eqs. (8.1) on the random vector of perturbations V and the initial conditions ΔX_0 that are random in the general case, the realizations of functions (8.7) are random.

Suppose the law of the distribution of vector $\Lambda = \{V, \Delta X_0\}$ composed of elements of vectors V and ΔX_0 is assigned and is defined by the function $f_\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_{m+n})$. Based on data of continuous measurements (8.7) over the interval $t \in [t_1, t_2]$, let us set up the problem of determining an estimate of some function $y(\Delta X, t_{pred}$ /prediction/) computed for the elements of the mismatch vector ΔX at the instant of time $t = t_{pred}$ by treating the measurements with the relation (see Fig. 8.1)

$$\hat{y}(\Delta X, t_n) = \sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, \eta_{li}(\Delta X(\tau), \tau)\, d\tau, \tag{8.8}$$

where $a_i(t)$ ($i = 1, 2, \ldots, l$) are determinate functions to be determined. /282

Eq. (8.8) can be represented in the form

$$\hat{y}(t_{pred}, \Lambda) = \sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, \eta_{li}(\Lambda, \tau)\, d\tau. \tag{8.9}$$

Since $y(t_{pred}, \Lambda)$ and $\hat{y}(t_{pred}, \Lambda)$ are functions of random vector Λ, the problem of determining the functions $a_i(t)$, ($i = 1, 2, \ldots, l$) can be formulated as a problem of minimizing the mathematical expectation of the square of the error in predicting the state $y(t_{pred}, \Lambda)$ by using Eq. (8.9)

$$(\Lambda) = y(t_{pred}, \Lambda) = y(t_{pred}, \Lambda). \tag{8.10}$$

Let us compute the criterion characterizing the precision of the prediction. We will have

$$I = M[\varepsilon^2(\Lambda)] = \int_{\Omega_\Lambda} \varepsilon^2(\Lambda) f_\Lambda(\Lambda)\, d\Lambda =$$

$$= \int_{\Omega_\Lambda} \left\{ y(t_{pred}, \Lambda) - \sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, \eta_i(\Lambda, \tau)\, d\tau \right\}^2 f_\Lambda(\Lambda)\, d\Lambda, \qquad (8.11)$$

where Ω_Λ is a set of realizations of random factors Λ.

After uncomplicated transformations of Eq. (8.11), we get

$$I = M[y^2(t_{pred}, \Lambda)] - 2\sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, M[y(t_{pred}, \Lambda)\, \eta_i(\Lambda, \tau)]\, d\tau + $$
$$+ \sum_{i,j=1}^{l} \int_{t_1}^{t_2}\int_{t_1}^{t_2} a_i(\tau)\, a_j(\gamma)\, M[\eta_i(\Lambda, \tau)\, \eta_j(\Lambda, \gamma)]\, d\tau\, d\gamma. \qquad (8.12)$$

Introducing the notation

$$R_y(t_{pred}) = M[y^2(t_{pred}, \Lambda)],$$
$$R_{y\eta_i}(t_{pred}, t) = M[y(t_{pred}, \Lambda)\ \ \eta_i(t, \Lambda)],$$
$$R_{\eta_i\eta_j}(t, \tau) = M[\eta_i(\Lambda, t)\ \ _j(\Lambda, \tau)],$$

let us write out Eq. (8.12) for the criterion of the estimate of the quality of precision in the form

$$I = R_y(t_{pred}) - 2\sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, R_{y\eta_i}(t_{pred}, \tau)\, d\tau +$$
$$+ \sum_{i,j=1}^{l} \int_{t_1}^{t_2}\int_{t_1}^{t_2} a_i(t)\, a_j(\tau)\, R_{\eta_i\eta_j}(t, \tau)\, dt\, d\tau. \qquad (8.13)$$

Let us find the necessary conditions for an extremum of the quality criterion (8.13). To do this, let us represent the desired functions $a_i(t)$, $(i = 1, 2, \ldots, l)$ in the form /283

$$a_i(t) = \tilde{a}_i(t) + \gamma_i \Delta a_i(t) \quad (i = 1, 2, \ldots, l), \qquad (8.14)$$

where γ_i are constant cofactors and $\Delta a_i(t)$ are arbitrary functions not identically equal to zero on the interval $t \in [t_1, t_2]$ and satisfying the following conditions:

$$\Delta a_i(t_1) = \Delta a_i(t_2) = 0 \quad (i = 1, 2, \ldots, l).$$

Substituting Eqs. (8.14) into the right-hand side of Eq. (8.13), we can easily obtain from the condition

$$\left.\frac{\partial I}{\partial \gamma_l}\right|_{\gamma_l=0}=0$$

the necessary conditions for determining functions $a_i(t)$, (i = 1, 2, ..., l) that are optimal in the sense of criterion (8.13), in the form of a system of linear integral Fredholm equations of the first kind:

$$R_{y\eta_r}(t_{\text{pred}},\ t)=\sum_{l=1}^{l}\int_{t_1}^{t_2}\tilde{a}_l(\tau)\,R_{\eta_r\eta_l}(t,\tau)\,d\tau \quad (r=1,2,\ldots,l). \tag{8.15}$$

We can easily show that conditions (8.15) are also sufficient conditions for a minimum for criterion (8.13).

Considering the fact that the functions (8.7) and the function $y(t_{\text{pred}})$ measured for the nonlinear process (8.1) are in the general case uncentered random functions with mathematical expectations $m_{\eta_1}(t), m_{\eta_2}(t),\ldots, m_{\eta_l}(t)$ and $m_y(t_{\text{pred}})$ and correlation functions $K_{\eta_i\eta_j}(t,\tau)$, (i, j = 1, 2, 3, ..., m), $K_{y\eta_i}(t_{\text{pred}},\ t)$ (i = 1, 2, ..., l), we can rewrite the system of integral equations (8.15) in the form

$$K_{y\eta_j}(t_{\text{pred}},\ t)+m_y(t_{\text{pred}})\,m_{\eta_j}(t)=$$
$$=\sum_{l=1}^{l}\int_{t_1}^{t_2}\tilde{a}_l(\tau)\left[K_{\eta_j\eta_l}(t,\tau)+m_{\eta_j}(t)\,m_{\eta_l}(\tau)\right]d\tau \quad (j=1,2,\ldots,l) \tag{8.16}$$

or

$$\sigma_y(t_{\text{pred}})\,\sigma_{\eta_j}(t)\,r_{y\eta_j}(t_{\text{pred}},\ t)+m_y(t_{\text{pred}})\,m_{\eta_j}(t)=$$
$$=\sum_{i=1}^{l}\int_{t_1}^{t_2}\tilde{a}_i(\tau)\left[\sigma_{\eta_j}(t)\,\sigma_{\eta_i}(\tau)\,r_{\eta_i\eta_j}(t,\tau)+\right.$$
$$\left.+\,m_{\eta_j}(t)\,m_{\eta_i}(\tau)\right]d\tau \quad (j=1,2,\ldots,l). \tag{8.17}$$

where $r_{\eta_i\eta_j}(t,\tau)$ are normalized correlation functions.

Let us look at several particular cases. /284

1. Suppose

$$K_{y\eta_j} = K_{\eta_i\eta_j} = 0 \quad (i, j = 1, 2, \ldots, l),$$

then Eqs. (8.17) become

$$m_y(t_{pred}) = \sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, m_{\eta_i}(\tau)\, d\tau. \tag{8.18}$$

2. Suppose $m_y(t_{pred}) = m_{\eta_i}(t) \equiv 0, \quad (i=1, 2, \ldots, l)$, then Eqs. (8.17) become:

$$\sigma_y(t_{pred})\, r_{y\eta_j}(t, \tau) = \sum_{i=1}^{l} \int_{t_1}^{t_2} \tilde{a}_i(\tau)\sigma_{\eta_i}(\tau)\, r_{\eta_j\eta_i}(t, \tau)\, d\tau \quad (j = 1, 2, \ldots, l). \tag{8.19}$$

Obviously, Eq. (8.18) does not enable us to find the desired functions $a_i(t)$. Eqs. (8.19) enable us to solve the problem of determining the functions $\tilde{a}_i(t)$; here the absence of a correlational reciprocity between the measured functions $\eta_i(t)$, (i = 1, 2, ..., l) enables us to obtain the system of independent integral Fredholm equations of the first kind for determining each of the desired functions $\tilde{a}_i(t)$, (i = 1, 2, ..., l) separately:

$$\sigma_y(t_{pred})\, r_{y\eta_j}(t_{pred}, t) = \int_{t_1}^{t_2} \tilde{a}_i(\tau)\, \sigma_{\eta_i}(\tau)\, r_{\eta_i\eta_j}(t, \tau)\, d\tau.$$

If the measurements (8.7) are made with the error

$$\delta\eta_1(t), \delta\eta_2(t), \ldots, \delta\eta_l(t)$$

and the latter is determined by two of the commonest cases:

$$\eta_i(t) = \tilde{\eta}_i(t) + \delta\eta_i(t),$$
$$\eta_i(t) = \tilde{\eta}_i(t)(1 + \delta\eta_i(t)) \quad (i = 1, 2, \ldots, l),$$

then the integral equations (8.15) can be written in the form:

$$R_{y\tilde{\eta}_j}(t_{pred}, t) = \sum_{i=1}^{l} \int_{t_1}^{t_2} \tilde{a}_i(\tau)\left[R_{\tilde{\eta}_j\tilde{\eta}_i}(t, \tau) + R_{\delta\eta_j\delta\eta_i}(t, \tau)\right] d\tau \quad (j = 1, 2, \ldots, l),$$

$$R_{y\tilde{\eta}_j}(t_{pred}, t) = \sum_{i=1}^{l} \int_{t_1}^{t_2} \tilde{a}_i(\tau)\, R_{\tilde{\eta}_j\tilde{\eta}_i}(t, \tau)\left[1 + R_{\delta\eta_j\delta\eta_i}(t, \tau)\right] d\tau \; (j = 1, 2, \ldots, l),$$

if the real instantaneous values of $\tilde{\eta}_i(t)$ are not correlated with a measurement error of $\delta\eta_i(t)$.

Using Eqs. (8.13) and (8.15), let us write out a formula for /285 computing the criterion of the estimate of the effectiveness of predicting the function $y(t_{pred})$, employing Eq. (8.8):

$$I = R_y(t_{pred}) - \sum_{i=1}^{l} \int_{t_1}^{t_2} \tilde{a}_i(\tau) R_{y\eta_i}(t_{pred}, \tau)\, d\tau. \qquad (8.20)$$

Eq. (8.20) enables us to estimate the error of prediction and to make a conclusion of the possibility of employing the prediction algorithm.

Of interest is an examination of the algorithm for the nonlinear treatment of observation results. To derive the working relations, it is sufficient to assume that

$$\eta_1(\Delta X, t) = z(t),\ \eta_2(\Delta X, t) = z^2(t),\ \ldots,\ \eta_l(\Delta X, t) = z^l(t),$$

where z(t) is the observation function.

Then with reference to Eq. (8.15) we will have the following system of linear integral Fredholm equations of the first kind

$$M[y(t_{pred}) z^j(t)] = \sum_{i=1}^{l} \int_{t_1}^{t_2} \tilde{a}_i(\tau) M[z^j(t) z^i(\tau)]\, d\tau \quad (j = 1, 2, \ldots, l). \qquad (8.21)$$

In predicting the function $y(t_{pred})$ based on one observation function, we obtain a linear Fredholm equation of the first kind

$$R_{y\eta_i}(t_{pred}, t) = \int_{t_1}^{t_2} \tilde{a}(\tau) R_{\eta\eta_i}(t, \tau)\, d\tau. \qquad (8.22)$$

Above we examined the problem of predicting process states based on data of instantaneous measurements in a fairly general case, namely: in the absence of any assumptions on the process $\Delta X(t)$. In particular cases, the integral equations (8.15) can be considerably simplified.

A fair number of studies /19, 32, 60/ deal with methods of solving the resulting linear integral equations of the Fredholm type (8.15). If $R_{\eta\eta}(t, \tau)$ in Eq. (8.22) is a symmetric quadratically-summable positive-determinate kernel and if Eq. (8.22) is solvable, for its solution we can employ the method of successive approximations.

We can also use the Fredholm equations in /32/ and convert from integral equation (8.22) to a system of algebraic equations. To do this, we must subdivide the interval (t_1, t_2) into n equal intervals having the length

$$\Delta\tau = \Delta t = \frac{t_2 - t_1}{n}$$

and we must set: /286

$$R_{\eta\eta}(t_1 + p\Delta t,\ t_1 + q\Delta\tau) = R^{pq}_{\eta\eta} \quad (p, q = 1, 2, \ldots, n),$$
$$R_{y\eta}(t_{\text{pred}},\ t_1 + p\Delta t) = R^{p}_{y\eta} \quad (p = 1, 2, \ldots, n),$$
$$a(t_1 + q\Delta\tau) = a^q \quad (q = 1, 2, \ldots, n).$$

Substituting for the integral $\int_{t_1}^{t_2} a(\tau) R_{\eta\eta}(t, \tau)\, d\tau$, when t = t_1 + p t, the sum

$$\sum_{q=1}^{n} R^{pq}_{\eta\eta} a^q \Delta\tau \quad (p = 1, 2, \ldots, n),$$

we get in place of integral equation (8.22) a system of linear algebraic equations

$$\sum_{q=1}^{n} R^{pq}_{\eta\eta} a^q \Delta\tau = R^{p}_{y\eta} \quad (p = 1, 2, \ldots, n). \tag{8.23}$$

If the determinant $|R^{pq}_{\eta\eta}|$, composed of the elements $R^{pq}_{\eta\eta}$, is not equal to zero, the system of equations (8.23) has a unique solution for any values of $R^{p}_{y\eta}$(p = 1, 2, ..., n) not identically equal to zero. By virtue of the fact that $R_{\eta\eta}(t, \tau) \leqslant R_{\eta\eta}(t) \geqslant 0$ and the symmetry of the function $R_{\eta\eta}(t, \tau)$, the determinant is not always equal to zero and the solution of the integral equation does exist. However, by virtue of the arbitrariness of the random process $\Delta X(t)$ and the approximate computation of the statistical characteristics of the nonlinear process (8.1), when solving the integral equation we can encounter the poor causality of matrix $(R^{pq}_{\eta\eta})$. Employing the methods of solving systems of linear algebraic equations, we can find the desired solutions -- the function a(t) specified at discrete points.

We can also proceed analogously with the problem of solving the system of integral equations (8.15). In this case we will be dealing with the matrix

$$D=\begin{pmatrix} R_{\eta_1\eta_1}(t,\tau) & R_{\eta_2\eta_1}(t,\tau) & \dots & R_{\eta_l\eta_1}(t,\tau) \\ R_{\eta_1\eta_2}(t,\tau) & R_{\eta_2\eta_2}(t,\tau) & \dots & R_{\eta_l\eta_2}(t,\tau) \\ \dots & \dots & \dots & \dots \\ R_{\eta_1\eta_l}(t,\tau) & R_{\eta_2\eta_l}(t,\tau) & \dots & R_{\eta_l\eta_l}(t,\tau) \end{pmatrix}$$

and with the vectors $A^*(t)=(a_1(t)a_2(t)\dots a_l(t))$,

$$B^*(t)=(R_{y\eta_1}(t_{pred},t)\; R_{y\eta_2}(t_{pred},t)\dots R_{y\eta_l}(t_{pred},t)),$$

enabling us to write out the integral equation (8.15) in matrical form

$$B(t_{pred},t)=\int_{t_1}^{t_2} D(t,\tau)A(\tau)\,d\tau. \tag{8.24}$$

Subdividing, as before, the interval $[t_1, t_2]$ into n equal /287 parts and carrying out transformations analogous to those examined above for the one-dimensional case, we get the matrix (D^{pq}) of of order (n, n), whose elements will be square matrices D^{pq} of order (l, l), whose inversion then gives the desired solution. However, in the multidimensional case of the integral equation (8.24), considerable difficulties can arise in the inversion of matrices of order [(n, l), (n, l)] when there are large values of n and l. In addition, we can also observe phenomena of the poor causality of matrix (D^{pq}) owing to the presence of a functional relation between the cross sections of individual measurement functions. Therefore in setting up the problem of prediction we must take account of the specific features of the process under study, and when there is poor causality of matrices we must modify either the observation interval or we must reject measurements not enabling us to predict the future states of the process.

When examining the problem of predicting $y(t_{pred})$ based on data of observation of the functions $\eta_i(t)$, it was assumed that the functions $\eta_i(t)$ include errors of measurement $\delta\eta_i(t)$, whose filtration was not carried out during the measurement. In principle we can set up and solve the problem of prediction for the case when the filtration is carried out in advance for the observation functions $\eta_i(t)$ and carry out the prediction of $y(t_{pred})$ based on the output signals of the filter $f_i(t)$. All working formulas here remain as before, and only the procedure of computing the statistical characteristics of the observation functions required for prediction will change. Let us examine this process more closely in order to find out in what relation we will consider the problems of filtration and prediction. Using the dynamic model of filtration and prediction shown in Fig. 8.2, let us write out the quality criterion for estimating the precision of prediction, here considering that

$$f_i(t) = \int_0^t w_i(t, \tau)\, \nu_i(\tau)\, d\tau,$$

$$I = R_y(t_{pred}) - 2\sum_{i=1}^{l} \int_{t_1}^{t_2} a_i(\tau)\, R_{yf_i}(t_{pred}, \tau)\, d\tau + \\ + \sum_{i,j=1}^{l} \int_{t_1}^{t_2}\int_{t_1}^{t_2} a_j(\tau)\, a_i(\tau)\, R_{f_i f_j}(t, \tau)\, dt\, d\tau. \tag{8.25}$$

Since

$$R_{yf_i}(t_{pred}, \tau) = M\left[y\ (t_{pred}) \int_0^\tau w_i(\tau, \lambda)\, \nu_i(\lambda)\, d\lambda\right] = \int_0^\tau w_i(\tau, \lambda)\, R_{y\nu_i}(t_{pred}, \lambda)\, d\lambda,$$

$$R_{f_i f_j}(t, \tau) = \int_0^t \int_0^\tau w_i(t, \delta)\, w_j(\tau, \lambda)\, R_{\nu_i\nu_j}(\delta, \lambda)\, d\delta\, d\lambda,$$

then we will have /288

$$I = R_y(t_{pred}) - 2\sum_{i=1}^{l} \int_{t_1}^{t_2}\int_0^\tau a_i(\tau)\, w_i(\tau, \lambda)\, R_{y\nu_i}(t_{pred}, \lambda)\, d\lambda\, d\tau + \\ + \sum_{i,j=1}^{l} \int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_0^t\int_0^\tau a_i(t)\, a_j(\tau)\, w_i(t, \delta)\, w_j(\tau, \lambda)\, R_{\nu_i\nu_j}(\delta, \lambda)\, dt\, d\tau\, d\delta\, d\lambda. \tag{8.26}$$

From the condition that criterion (8.26) is a minimum, we find the required optimality conditions for finding functions $w_i(t, \tau)$ and $a_i(t)$.

Introducing into consideration the functions

$$w_i(t, \tau) = \tilde{w}_i(t, \tau) + p_i \Delta w_i(t, \tau),$$

$$a_i(t) = \tilde{a}_i(t) + q_i \Delta a_i(t) \quad (i = 1, 2, \ldots, l)$$

and substituting them into criterion (8.26), from the conditions

$$\left.\frac{\partial I}{\partial p_k}\right|_{p_k=q_k=0}=0,$$

$$\left.\frac{\partial I}{\partial q_k}\right|_{p_k=q_k=0}=0 \quad (k=1, 2, \ldots, l)$$

we obtain the following relations:

$$\frac{\partial I}{\partial p_k}=0: \int_{t_1}^{t_2}\int_0^{\tau}\Delta w_k(\tau, \lambda)\left[\tilde{a}_k(\tau)\left\{R_{y\nu_k}(t_{\text{pred}}, \lambda)-\int_{t_1}^{t_2}\int_0^{t}\sum_{j=1}^{l}\tilde{a}_j(t)\times\right.\right.$$

$$\left.\left.\times\tilde{w}_j(t, \delta)R_{\nu_k\nu_j}(\delta, \lambda)\,d\delta dt\right\}\right]d\tau d\lambda=0 \quad (k=1, 2, \ldots, l);$$

$$\frac{\partial I}{\partial q_k}=0: \int_{t_1}^{t_2}\Delta a_k(\tau)\left[\int_0^{\tau}\tilde{w}(\tau, \lambda)\left\{R_{y\nu_k}(t_{\text{pred}}, \lambda)-\int_{t_1}^{t_2}\int_0^{t}\tilde{a}_j(t)\,\tilde{w}_j\times\right.\right.$$

$$\left.\left.\times(t, \delta)R_{\nu_k\nu_j}(\delta, \lambda)\,d\delta dt\right\}\right]d\lambda d\tau=0 \quad (k=1, 2, \ldots, l),$$

whence it follows that the necessary conditions coincide and are of the form

$$R_{y\nu_k}(t_{\text{pred}}, \lambda)=\sum_{j=1}^{l}\int_{t_1}^{t_2}a_j(t)\left(\int_0^{t}w_j(t, \delta)R_{\nu_k\nu_j}(\delta, \lambda)\,d\delta\right)dt \quad (k=1, 2, \ldots, l). \tag{8.27}$$

The foregoing shows that on the condition that criterion (8.26) is a minimum, we were unable to obtain the conditions necessary for finding either the optimal impulse transfer function $\tilde{w}_i(t, \tau)$ and the weighting function of the predicting device $\tilde{a}_i(t)$.

Introducing the notation /289

$$\mu_{jk}(t, \lambda)=\int_0^{t}\tilde{w}_j(t, \delta)R_{\nu_k\nu_j}(\delta, \lambda)\,d\delta, \tag{8.28}$$

let us rewrite condition (8.27) in the form

$$R_{y\nu_k}(t_{\text{pred}}, \lambda)=\sum_{i=1}^{l}\int_{t_1}^{t_2}a_j(t)\mu_{ij}(t, \lambda)\,dt \quad (k=1, 2, \ldots, l). \tag{8.29}$$

Comparing conditions (8.15) and (8.29), we can note that they are equivalent in form. Essentially, the very conditions (8.29) enable us to find the solution of the problem formulated only given the condition that we have the known impulse transfer function $w_j(t, \tau)$ of the dynamical system enabling us to filtrate the signal measured. In the one-dimensional case we will have

$$\mu(t, \lambda) = \int_0^t w(t, \vartheta) R_{\nu\nu}(\vartheta, \lambda)\, d\vartheta, \tag{8.30}$$

$$R_{y\nu}(t_{pred}, \lambda) = \int_{t_1}^{t_2} a(t) \mu(t, \lambda)\, dt. \tag{8.31}$$

We can easily see that conditions (8.30) essentially are Weiner conditions, that is, the condition of the optimal filtration of the signal measured, therefore in the one-dimensional case the separate solution of the sequence of problems in optimal filtration and prediction of the states of the random process ensure an optimal value of the quality criterion (8.26), since

$$\mu(t, \lambda) = R_{f\nu}(t, \lambda).$$

Thus, Eq. (8.31) is of the form

$$R_{y\nu} = (t_{pred}, \lambda) = \int_{t_1}^{t_2} a(\tau) R_{f\nu}(\tau, \lambda)\, d\tau. \tag{8.32}$$

Comparing Eqs. (8.15) and (8.32), we can note that essentially they differ in the correlation in the right-hand side of $R_{f\nu}(\tau, \lambda)$ and $R_{\nu\nu}(\tau, \lambda)$, which naturally is important from the standpoint of solving the prediction problem.

Above we examined the algorithm of the continuous treatment of observation functions for the purpose of solving the problem of predicting the future states. The operating scheme of the prediction device is given in Figs. 8.1 and 8.2.

When digital computers are used for solving the problem of predicting the future states of a process, the observation functions $\eta_i(t)$ will be measured at the discrete points $t_1 + t, \ldots, t_2$ /290
with an interval Δt:

$$\begin{array}{l}
\eta_1(t_1 + \Delta t),\ \eta_1(t + 2\Delta t), \ldots, \eta_1(t_1 + j\Delta t), \ldots, \eta_1(t_2), \\
\eta_2(t_1 + \Delta t),\ \eta_2(t + 2\Delta t), \ldots, \eta_2(t_1 + j\Delta t), \ldots, \eta_2(t_2), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\eta_l(t_1 + \Delta t),\ \eta_l(t + 2\Delta t), \ldots, \eta_l(t_1 + j\Delta t), \ldots, \eta_l(t_2)
\end{array}$$

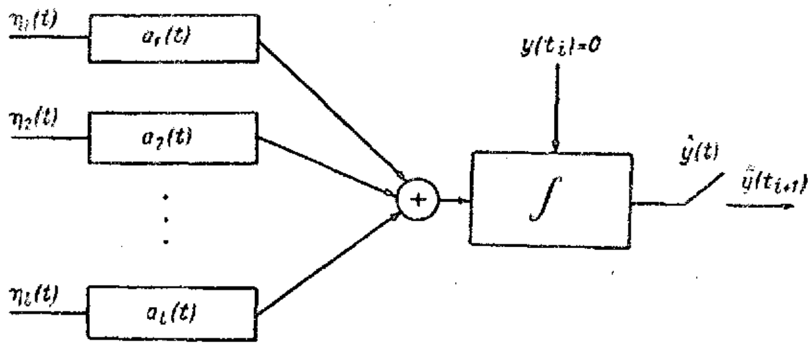


Fig. 8.1. Scheme for the prediction of phase coordinates at a given instant of time

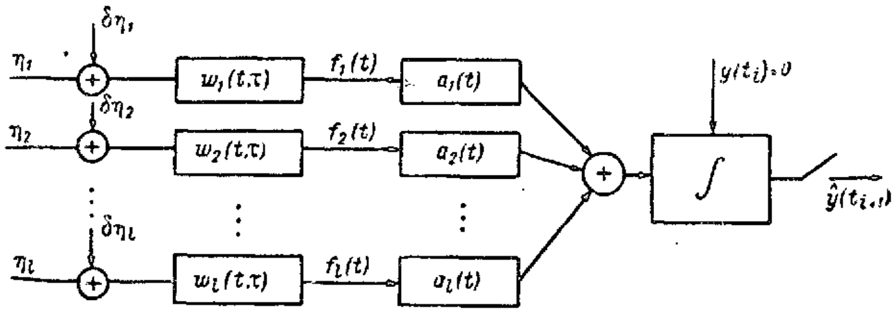


Fig. 8.2. Scheme of predicting phase coordinates at a given instant of time with smoothing of measured signals

and the problem of constructing the prediction algorithm reduces to the realization of the algorithm

$$\hat{y}(t_{pred}) = \sum_{i=1}^{l}\sum_{j=1}^{l} a_{ij}\eta_i(t_1 + j\Delta t). \qquad (8.33)$$

To determine the coefficients a_{ij}, let us introduce the vector A composed of elements a_{ij} of the form

$$A^* = (a_{11}a_{21}\ldots a_{l1}a_{12}a_{22}\ldots a_{ll}),$$

and let us arrange the measurements of the observation functions in a vector-row of the form /291

$$C^* = (\eta_1(\Delta t)\,\eta_2(\Delta t)\ldots\eta_l(\Delta t)\,\eta_1(2\Delta t)\ldots\eta_l(t_l)).$$

Then Eq. (8.33) can be represented in the form

$$\hat{y}(t_{pred}) = \sum_{l=1}^{N} c_l a_l, \quad N = l \times n.$$

Formulating the criterion for the estimate of the precision of prediction, in the form

$$I = \int_{\Omega_\Lambda} \left[y(t_{pred}, \Lambda) - \sum_{l=1}^{N} c_l(\Lambda) a_l \right]^2 f_\Lambda(\Lambda)\, d\Lambda,$$

we obtain a system of linear algebraic equations for determining the coefficients

$$M[y(t_{pred})\, c_r] = \sum_{l=1}^{N} a_l M[c_l c_r] \quad (r = 1, 2, \ldots, N). \tag{8.34}$$

The expressions $M[y(t_{pred})\, c_r]$ and $M[c_l c_r]$ in the system of equations (8.34) denote $M[y(t_{pred})\, \eta_\nu(j\Delta t)]$ and $M[\eta_\nu(j\Delta t)\, \eta_\delta(i\Delta t)]$ for $\nu, \delta = 1, 2, \ldots, l$; $i, j = 1, 2, \ldots, n$.

In practical problems, l and n can be fairly large numbers, therefore in several cases it appears expedient to formulate the coefficients $a_i(t)$ by using some system of specified functions

$$\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t),$$

then we will have

$$a_i(t) = \sum_{j=1}^{M_1} a_{ij} \varphi_j(t) \quad (i = 1, 2, \ldots, l). \tag{8.35}$$

Substituting Eq. (8.35) into the equation for the quality criterion (8.13), we get

$$I = R_y(t_{pred}) - 2 \sum_{l=1}^{l} \int_{t_1}^{t_2} \sum_{k=1}^{M_1} a_{lk} \varphi_k(\tau) R_{y\eta_l}(t_{pred}, \tau)\, d\tau +$$

$$+ \sum_{l, j=1}^{l} \sum_{k, p=1}^{M_1} \int_{t_1}^{t_2} \int_{t_1}^{t_2} a_{lk} \varphi_k(\tau)\, a_{lp} \varphi_p(t)\, R_{\eta_l \eta_j}(t, \tau)\, d\tau =$$

$$= R_y(t_{pred}) - 2 \sum_{l=1}^{l} \sum_{k=1}^{M_1} a_{lk} w_k^l + \sum_{l, j=1}^{l} \sum_{k, p=1}^{M_1} a_{lk} a_{jp}\, w_{cr}^{ij} \tag{8.36}$$

where we use the notation /292

$$w_k^i = \int_{t_1}^{t_2} R_{y\eta_i}(t_{pred}, \tau)\varphi_k(\tau)\,d\tau,$$

$$w_{cr}^{ij} = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \varphi_k(\tau)\varphi_p(t) R_{\eta_i\eta_j}(t,\tau)\,d\tau dt. \tag{8.37}$$

From the condition $\frac{\partial I}{\partial a_{ik}}=0$, we get the system of algebraic equations for determining the desired coefficients a_{ik}:

$$\sum_{j=1}^{l}\sum_{p=1}^{M_1} w_{cr}^{ij} a_{jp} = w_k^i \quad (i=1,2,\ldots,l;\ k=1,2,\ldots,M_1). \tag{8.38}$$

This procedure enables us to avoid the need to obtain a system of integral Fredholm equations of the first kind that have considerable inconveniences in its solution, however the convergence of the solution to the optimal value for a small number M_1 is difficult to ensure, although as $M_1 \to \infty$ the solutions (8.38) tend to $\tilde{a}_i(t)$.

For i = 1, the system of equations becomes

$$\sum_{p=1}^{M_1} w_{cr}\, a_p = w_k \quad (k=1,2,\ldots,M_1).$$

The mathematical expectations $M[y(t_{pred})\eta_i(t)]$ and $M[\eta_i(t)\eta_i(\tau)]$ necessary for carrying out the computations, and the elements w_{cr}^{ij}, w_k^i can be computed by processing the sequences

$$y(t_{pred}, \Lambda^{(1)}),\ y(t_{pred}, \Lambda^{(2)}),\ \ldots,\ y(t_{pred}, \Lambda^{(N)}),$$
$$\eta_{\eta_i}(t_{pred}, \Lambda^{(1)}),\ \eta_{\eta_i}(t_{pred}, \Lambda^{(2)}),\ \ldots,\ \eta_{\eta_i}(t_{pred}, \Lambda^{(N)}) \quad (i=1,2,\ldots,l), \tag{8.39}$$

obtained by integrating the system of differential equation (8.1) for the sequence of the random vector

$$\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(N)} \tag{8.40}$$

using the digital computer. Difficulties associated with computing the coefficients w_{cr}^i and w_k^i based on Eqs. (8.37) can be overcome to a large extent if these coordinates are represented in the form

$$w_k^i = M\left[z_k^i y\,(t_{pred})\right],$$

$$w_{cr}^{ij} = M\left[z_k^i z_p^j\right],$$

where

$$z_k^i = \int_{t_1}^{t_2} \eta_i(t)\,\varphi_k(t)\,dt. \qquad (8.41)$$

Realization (8.41) can be easily computed on a digital computer by modeling the process (8.1), if the initial system of equations is supplemented with a system of differential equations /293

$$\dot{z}_k^i = \eta_i(t)\,\varphi_k(t),\; t \in [t_1, t_2]\; (i = 1, 2, \ldots, l;\; k = 1, 2, \ldots, M_1).$$

For the remainder, the treatment of the sequences

$$z_k^{(i)}(\Lambda^{(1)}),\; z_k^{(i)}(\Lambda^{(2)}), \ldots,\; z_k^{(i)}(\Lambda^{(N)}),$$

computed for the sequences (8.40) can be carried out by using recursion relations of the method of statistical tests.

In solving the problem of constructing the prediction algorithm, we used a priori information on perturbing actions causing the observed process to deviate from the reference process. The control process does not provide for the accumulation of information on the perturbing actions and adaptation of the resulting algorithm.

Use of prediction results in accordance with problems 8.2 and 8.3, for optimal control of a process, requires that we know the effectiveness of the control actions applied at the instants of correction t_1, t_2, ..., t_p for the phase states of the process at subsequent instants of correction and at the terminal state of the system.

8.3. Determination of the Effectiveness of Control Actions on the State of the Process

In formulating the optimal control $\Delta U_1, \Delta U_2, \ldots, \Delta U_p$, we must determine the state of the process that is caused by the given control, that is, we must know the structure and parameters of the state of the system under study as a function of the controls ΔU_i

$$\Delta X(T) = \Delta X(\Delta U_1, \Delta U_2, \ldots, \Delta U_p, T) \qquad (8.42)$$

or the state of the system at the next instant of correction

$$\Delta X(t_{i+1}) = \Delta X(\Delta U_i, t_{i+1}). \qquad (8.43)$$

Since the controls ΔU_i are applied to system (8.1) discontinuously at the instant of time t_i and remain constant over the interval $t \in [t_i, t_{i+\Delta T}]$, determining the structure of functions (8.42) or (8.43) can be associated either with representing the latter with the Taylor series

$$\Delta X(T) = \Delta X(T, \Delta U_i = 0) + \sum_{i=1}^{p} \frac{\partial \Delta X(T)}{\partial \Delta U_i}\bigg|_0 \Delta U_i + \ldots, \qquad (8.44)$$

$$\Delta X(t_{i+1}) = \Delta X(t_{i+1}, \Delta U_i = 0) + \frac{\partial X(t_{i+1})}{\partial \Delta U_i}\bigg|_0 \Delta U_i + \ldots \qquad (8.45)$$

or with the approximation $\Delta X(T)$ and $\Delta X(t_i)$ by several polynomials: /294

$$\Delta X(T) = a_0 + \sum_{i=1}^{p} a_i \Delta U_i + \ldots, \qquad (8.46)$$

$$\Delta X(t_{i+1}) = a_0 + a_i \Delta U_i + \ldots. \qquad (8.47)$$

To construct series (8.44) and (8.45), it is necessary to compute for the reference motion of process (8.1) sensitivity functions of the first, second, and higher orders of the vector ΔX for the control actions.

Using sensitivity equations usually involves considerable preliminary work in determining differential sensitivity equations. First-order partial derivatives can be computed when there is a linear model of the process (8.1)

$$\Delta \dot{X} = A(t)\Delta X + B(t)\Delta U,$$

under study is available if the integral of convolution is used:

$$\Delta X_i(t) = \int_0^t w_i(t, \tau) B(\tau) \Delta U_i(\tau)\, d\tau.$$

Since $\Delta U_i = \mathrm{const}$ over the interval $t \in [t_i, t_{i+\Delta T}]$, then we will have

$$\frac{\partial \Delta X}{\partial \Delta U_i}(t_{i+1}) = \int_{t_i}^{t_{i+1}} w_i(t_{i+1}, \tau) B(\tau)\, d\tau,$$

$$\frac{\partial \Delta X}{\partial \Delta U_i}(T) = \int_{t_i}^{T} w_i(T, \tau) B(\tau)\, d\tau. \qquad (8.48)$$

For calculations based on Eqs. (8.48), it is necessary first to compute the cross sections $w(t_1, t)$, $w(T, t)$ of the impulse transfer function $w(t, \tau)$ by the linear model $(\Delta\dot{X} = A\Delta X)$ at different instants of time $t_1, t_2, \ldots, t_p, T$.

In principle, both these approaches can be successfully used in solving the problem formulated, however they are marked by one common disadvantage, associated with considerable preliminary work in linearizing the system of nonlinear differential equations (8.1). One should note an additional circumstance that often is not given appropriate attention in constructing the control for nonlinear processes. When reference trajectories are computed often equations (8.1) are integrated under the condition that all the random components of the perturbing factors are equal to zero, that is, the system of equations

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, [M v_1], \ldots, M(v_m), \tilde{U}(t)),$$
$$x_i(t_0) = M|x_{i,0}| \quad (i = 1, 2, \ldots, n). \tag{8.49}$$

is investigated.

We denote solutions to the system of equations (8.49) by $\tilde{x}_i(t)$ $(i = 1, 2, \ldots, n)$. Let us denote the mathematical expectations of the solutions to system of equations (8.1) by $\bar{x}_i(t) = M[x_i(t)]$, $(i = 1, 2, \ldots, n)$. Obviously, the difference of the resulting solution /295

$$\Delta\tilde{x}_i(t) = \bar{x}_i(t) - \tilde{x}_i(t) \quad (i = 1, 2, \ldots, n) \tag{8.50}$$

is not equal to zero in the general case.

Control of the process (8.1) in order to reduce the mismatches

$$\Delta x_i(t, V) = x_i(t, V) - \tilde{x}_i(t) = \Delta\tilde{x}_i(t) + \Delta x_i(t, V) \quad (i = 1, 2, \ldots, n) \tag{8.51}$$

to zero involves compensating both the systematic error $\Delta\tilde{x}_i(t)$, as well as the random component of the mismatches, which naturally can lead to considerable deviations of the control actions from the reference values and to considerable deviations of the instantaneous states of the process from the reference states when there are constraints on the control actions.

To compensate the statistical error of mismatches $\Delta\tilde{x}_i(t)$, it is obvious that we must find the mathematical expectations of the correcting impulses ΔU_i so as to ensure $\Delta\tilde{x}_i(t)=0$, or to consider as programming motions the phase states $\bar{x}_i(t)$, which in general gives the same effect, and namely there is a replacement of the reference motion with respect to phase coordinates $\tilde{x}_i(t)$ by the phase coordinates $\bar{x}_i(t)$ $(i = 1, 2, \ldots, n)$. The latter means that linearization of the equations of motion (8.1) must be conducted so that the elements of the matrices of the linear model of the process

$$\frac{d\Delta\mathring{x}_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\Delta\mathring{x}_j + \sum_{j=1}^{r} b_{ij}\Delta U_j + \sum_{j=1}^{m} c_{ij}V_j \tag{8.52}$$

are computed for the reference trajectory characterized by the state $\bar{x}_i(t)$ $(i = 1, 2, \ldots, n)$, that is,

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_i=\bar{x}_i(t)};\quad b_{ij} = \left.\frac{\partial f_i}{\partial \Delta U_j}\right|_{x_i=\bar{x}_i(t)};\quad c_{ij} = \left.\frac{\partial f_i}{\partial V_j}\right|_{x_i=\bar{x}_i(t)}.$$

From the computational point of view, this means that in order to obtain the phase coordinates $\bar{x}_i(t)$ it is necessary to investigate the nonlinear perturbed system of equations (8.1) by one of the methods of analyzing the scatter of nonlinear stochastic systems for determining the solutions $\bar{x}_i(t)$, and then to linearize the nonlinear equations.

Use of sensitivity equations also presupposes the parametric linearization of nonlinear stochastic equations (8.1) relative to the solutions $\bar{x}_i(t)$: /296

$$\dot{z}_{ik} = \sum^{n} a_{il}z_{lk} + b_{ik}(t) \quad (k = 1, 2, \ldots, p), \tag{8.53}$$

where

$$z_{ik} = \frac{\partial x_i}{\partial \Delta U_k},\quad a_{il} = \left.\frac{\partial f_i}{\partial x_l}\right|_{x_i=\bar{x}_i},$$

$$b_{ik} = \left.\frac{\partial f_i}{\partial \Delta U_k}\right|_{x_i=\bar{x}_i}.$$

Therefore to compute the effectiveness of control actions by means of the system of linearized equations (8.52) or the sensitivity models (8.53) requires considerable preliminary work in determining the solutions $\overline{x}_i(t)$ and in linearizing nonlinear equations (8.1) with respect to the solutions found.

Accordingly, it appears possible to indicate a less laborious algorithm of computing the effectiveness of control actions by employing the above-described method of stochastic approximation.

Using the working formulas obtained in Chapter Six and computing the elements of vector Z with one of the methods of investigating nonlinear systems, we obtain the structure of the function ΔX with respect to the control parameters. As a whole, the computational procedure consists of the following.

We introduce the vector of random parameters $\mu=(\Delta U_1, \Delta U_2, \ldots, \Delta U_p)$; we determine the range of the variation in the control parameters; and we assume that they are random with a normal or uniform distribution and that there is no correlation between the elements of the vector μ. Then we determine the statistical characteristics of vector μ based on a knowledge of the range of variation of the controls. We construct a random sequence of vectors

$$\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(N)} \tag{8.54}$$

and for each of its elements, by solving the system of differential equations (8.1) for the vector $\Lambda=0(V=0, \Delta\dot{X}_0=0)$, we construct a sequence of vectors $\Delta X(t_1), \Delta X(t_2), \ldots, \Delta X(t_p), \Delta\dot{X}(T)$, for example

$$\Delta X^{(1)}(T), \Delta X^{(2)}(T), \ldots, \Delta X^{(N)}(T). \tag{8.55}$$

Treatment of the sequences (8.54) and (8.55) enables us to compute the elements of the vector $Z_i^p = M[\Delta X(T)\Delta U_i^p]$, and this means, the desired regression coefficients.

Naturally, when the functions $\Delta X(t_i)$ are approximated by /297 polynomials, the question of the degree of the approximating polynomial arises. This question can be answered by computing successively regression coefficients for two neighboring degrees of polynomials or from an analysis of the error of approximation.

8.4. Optimal Controls with Prediction of Phase Coordinates

Let us examine successively the solution to problems 8.1-8.3 formulated in Section 8.1. In problem 8.1 there is no problem of predicting future states, however it quite closely borders in structure of control algorithm problems 8.2 and 8.3 and its solution can be of interest from the standpoint of clarifying the advantages of optimal control with prediction of phase states.

Since seeking for the structure of the function $\phi(\Delta X)$ by using analytic methods of synthesis is a problem that is quite laborious and unresolvable in practice, the solution (8.4) is usually sought for in the class of specified functions by using the representation of the function $\phi(\Delta X)$ employing series of the form

$$\varphi\left[\Delta X(t_i)\right] = k_0^i + \sum_{j=1}^{n} k_j^{(i)} \Delta x_j(t_i) + \ldots, \tag{8.56}$$

whose coefficients are selected from the condition of ensuring that the quality criterion (8.3) has a minimum.

By introducing the one-dimensional vector q composed of the coefficients $k_0^i, k_j^{(i)}, k_{j\nu}^{(i)}, \ldots \ (i=1, 2, \ldots, p;\ \nu,\ j=1,\ 2,\ \ldots,\ p_1)$ and by substituting Eqs. (8.4) and (8.50) into system of equations (8.1), we get

$$\begin{gathered}
\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_m, \\
q_1, q_2, \ldots, q_s), \\
x_i(t_0) = \tilde{x}_{i,0} + \Delta x_{i,0} \quad (i = 1, 2, \ldots, n).
\end{gathered} \tag{8.57}$$

Thus, the control process is described by the system of non-linear differential equations (8.57), the right-hand sides of which contain the unknown parameters of the control algorithm. The numerical values of these parameters usually are found by solving the subsequent problem.

Problem 8.4. For the process described by the system of nonlinear stochastic equations (8.57), it is necessary to determine the parameters of control K on the condition that the quality criterion (8.3) is provided with a minimum value.

Problem 8.4 belongs to the class of problems of searching for an extremum of an implicit function of many variables and can be solved by one of the iterative methods of searching for an extremum discussed earlier. Here a good initial approximation for elements of the vector $q^{(0)}$ to a considerable extent determines the convergence of the iterative methods of searching for an extremum of criterion (8.3) and the volume of computations involved in optimizing it. Let us indicate therefore one of the possible procedures in selecting the initial approximation of vector $q^{(0)}$ for problems in control of the terminal state of process (8.1). /298

In problems of terminal control, usually the condition of ensuring at the final instant of time of operation of the system some set of conditions

$$\Delta L_i = L_i(X(t)) - L_i(\tilde{X}(t)) = 0 \quad (i = 1, 2, \ldots, \nu). \tag{8.58}$$

is set up.

Let us expand the functions ΔL_i in a Taylor series in elements of the vector ΔX:

$$\Delta L_i = \sum_{j=1}^{n} \frac{\partial L_i}{\partial x_j} \Delta x_j + L_{i,0}^i + \ldots \quad (i = 1, 2, \ldots, \nu). \tag{8.59}$$

If the number of conditions (8.58) is equal to the number of elements of the vector ΔU, that is, $p = \nu$, the coefficients of the Taylor series (8.61) computed for the reference motion at the correction instants of time can be taken as the zero approximation of the vector q with reference to the effectiveness of the control in the i-th instant of correction.

The problem of controlling the descent of a flight vehicle into the earth's atmosphere in the longitudinal and lateral planes by means of two controls -- the control of the angle of attack and the control of the slip angle -- can serve as an example.

If the problem of controlling the flight vehicle is to ensure that it lands at a specified point on the earth's surface, then naturally it is required to satisfy the two conditions

$$\Delta L_1 = L_1(X(t)) - L_1(\tilde{X}) = 0;$$
$$\Delta L_2 = L_2(X(t)) - L_2(\tilde{X}) = 0,$$

where L_1 and L_2 are the characteristics of the longitudinal and lateral motion of the flight vehicle.

By determining the expansion of the functions ΔL_1 and ΔL_2 in a Taylor series for the i-th control instants

$$\Delta L_1^i = \Delta L_{1,0}^i + \sum_{j=1}^{n} \frac{\partial \Delta L_1}{\partial x_j}\Big|_{t_i} \Delta x_j + \ldots,$$

$$\Delta L_2^i = \Delta L_{2,0}^i + \sum_{j=1}^{n} \frac{\partial \Delta L_2}{\partial x_j}\Big|_{t_i} \Delta x_j + \ldots. \tag{8.60}$$

and considering the effectiveness of the control (let us assume that it is determined by a linear regression)

$$\Delta\hat{L}_1^i(T, t_i) = C_{\Delta L_1}^i \Delta u_i^{(1)} + \gamma_{\Delta L_1}^i \Delta u_i^{(2)},$$
$$\Delta\hat{L}_2^i(T, t_i) = C_{\Delta L_2}^i \Delta u_i^{(1)} + \gamma_{\Delta L_2}^i \Delta u_i^{(2)},$$

from the conditions /299

$$\Delta L_1^{(i)} - \Delta\hat{L}_1^{(i)}(T, t_i) = 0,$$
$$\Delta L_2^{(i)} - \Delta L_2^{(i)}(T, t_i) = 0$$

we can obtain a control algorithm of the form

$$\Delta u_i^{(1)} = \frac{\gamma_{\Delta L_2}^i \Delta L_1^i - C_{\Delta L_2}^i \Delta L_2^i}{\Delta},$$
$$\Delta u_i^{(2)} = \frac{-\gamma_{\Delta L_1}^i \Delta L_1^i + C_{\Delta L_1}^i \Delta L_2^i}{\Delta}, \quad (8.61)$$

where

$$\Delta = C_{\Delta L_1}^i \gamma_{\Delta L_2}^i - C_{\Delta L_2}^i \gamma_{\Delta L_1}^i.$$

Substituting into Eqs. (8.61) Eqs. (8.60), we get the controls in the form

$$\Delta u_i^{(1)} = \frac{\gamma_{\Delta L_2}^i \Delta L_{1,0}^i - C_{\Delta L_2}^i \Delta L_{2,0}^i}{\Delta} + \frac{\gamma_{\Delta L_2}^i l^i - C_{\Delta L_2}^i b^i}{\Delta}\Delta X +$$
$$+ \Delta X^* \frac{ll^i \gamma_{\Delta L_2}^i - bb^i C_{\Delta L_2}^i}{\Delta}\Delta X + \ldots$$
$$\Delta u_i^{(2)} = \frac{-\gamma_{\Delta L_1}^i \Delta L_{1,0}^i + C_{\Delta L_1}^i \Delta L_{2,0}^i}{\Delta} + \frac{-\gamma_{\Delta L_1}^i l^i + C_{\Delta L_1}^i b^i}{\Delta}\Delta X +$$
$$+ \Delta X^* \frac{-\gamma_{\Delta L_1}^i ll^i + C_{\Delta L_1}^i bb^i}{\Delta}\Delta X + \ldots, \quad (8.62)$$

where

$$l^i = \left(\frac{\partial \Delta L_1}{\partial \Delta X}\right)_{t_i}, \quad ll^i = \left(\frac{\partial^2 \Delta L_1}{\partial \Delta X^2}\right)_{t_i};$$
$$b^i = \left(\frac{\partial \Delta L_2}{\partial \Delta X}\right)_{t_i}, \quad bb^i = \left(\frac{\partial^2 \Delta L_2}{\partial \Delta X^2}\right)_{t_i}.$$

From the equality of Eqs. (8.56) and (8.62) we get the zero approximation for the vector of coefficients $q^{(0)}$.

For the remainder, the solution to problem 8.4 is carried out by employing algorithms for the statistical optimization of the dynamic systems.

We can similarly solve problems of constructing the control for problems 8.2 and 8.3; here in the expansions of the control functions (8.56) we must use the values of the vector $\Delta X(t)$ obtained by prediction based on data of instantaneous measurements.

REFERENCES

/301

1. Anderson, T., Vvedeniye v mnogomernyy statisticheskiy analiz /Introduction to Multidimensional Statistical Analysis/, Moscow, Fizmatgiz, 1963.

2. Andreyev, N. I., Korrelyatsionnaya teoriya statisticheski optimal'nykh sistem /Correlational Theory of Statistically Optimal Systems/, Moscow, "Nauka", 1966.

3. Aref'yev, A. V. et al., "Results of Measuring Wind Regime in the Meteor Zone by Radar," Geomagnetizm i aeronomiya 6(4), (1966).

4. Bagrov, N. A., "Analytic Representation of a Sequence of Meteorological Fields by Eigenorthogonal Components," Trudy TsIP /Transactions of the Central Institute of Weather Forecasting/ (74), (1959).

5. Belyayev, V. P., Beltadze, T. G., and Litovchenko, V. L., "Results of Radiosonde Studies of Atmospheric Turbulence," Trudy TsAO /Transactions of the Central Aerological Observatory/ (54), (1964).

6. Bendat, Dzh., Osnovy teorii sluchaynykh shumov i yeye primeneniye /Essentials of the Theory of Random Noise and Its Applications/, Moscow, "Nauka", 1965.

7. Biryukov, L. A., and Kastrov, V. T., "Diurnal Trend of Temperature in the Stratosphere," Meteorologiya i gidrologiya (8), (1961).

8. Borovikova, A. S., and Mertsalova, O. B., "Latitudinal Root Mean Square Deviations of the Temperature of the Free Atmosphere over the Northern Hemisphere," Trudy NIIAK /Transactions of the Central Scientific Research Institute of Aeroclimatology/ (30), (1965).

9. Buslenko, N. P. et al., Metod statisticheskikh ispytaniy /Method of Statistical Tests/, Moscow, Fizmatgiz, 1962.

10. Vitel's, L. A., "Solar Activity, Transformation of Forms of Atmospheric Circulation, and Intramonthly Temperature Fluctuations," Trudy GGO /Transactions of the Main Geophysical Observatory imeni A. I. Voyeykov/ (87), (1959).

11. Vorob'yev, V. I., Vysotnyye frontal'nyye zony severnogo polushariya /Altitude Frontal Zones of the Northern Hemisphere/, Leningrad, Gidrometeoizdat, 1968.

12. Gantmakher, F. R., Teoriya matrits [Matrix Theory], Moscow, "Nauka", 1969.

13. Gorbatenko, S. A. et al., Mekhanika poleta [Flight Mechanics] Moscow, "Mashinostroyeniye", 1969.

14. Gnedenko, B. V., Kurs teoriya veroyatnostey [Course on Probability Theory], Moscow, "Nauka", 1969.

15. Delov, I. A., "Turbulent Motions in the Upper Atmosphere at Altitudes 90-110 km and Their Relationship With Other Phenomena," Problemy kosmicheskoy fiziki (1), (1966).

16. Delov, I. A., "Main Features of the Temperature Distribution in the Atmosphere in Different Seasons in the 0-100 km Layer," Meteorologiya i gidrologiya (9), (1968).

17. Zabreyko, P. P. et al., Integral'nyye uravneniya [Integral Equations], Moscow, "Nauka", 1968.

18. Ivanovskiy, A. I., and Kivganov, A. F., "Role of Radiative and Turbulent Inflows of Heat in Forming Temperature Stratification in the Stratosphere and Lower Mesosphere," Trudy TsAO (96), (1970).

19. Kazakov, I.Ye., and Dostupov, B. G., Statisticheskaya dinamika nelineynykh avtomaticheskikh sistem [Statistical Dynamics of Nonlinear Automatic Systems], Moscow, Fizmatgiz, 1962.

20. Kazakhov, I. Ye., Statisticheskiye metody proyektirovaniya sistem upravleniya [Statistical Metods of Designing Control Systems], Moscow, "Nauka", 1969.

21. Kalman, R., "Variational Principle of Selecting the Optimal Filter on the Condition of Minimum Squares of Error," in the collection: Samonastraivayushchiye avtomaticheskiye sistemy, Trudy Mezhdunarodnogo simpoziuma [Self-adjusting Automatic Systems, Transactions of the International Symposium], Moscow, "Nauka", 1964. /302

22. Kantorovich, L. V., and Krylov, V. I., Priblizhenyye metody vysshego analiza [Approximation Methods of Higher Analysis], Leningrad, Fizmattiz, 1962.

23. Kashcheyev, B. L., and Delov, I. A., "Streams in the Earth Atmosphere at the Altitudes 90-100 km," DAN USSR 7, (1964).

24. Kats, A. L., Tsirkulyatsiya v stratosfere i mezosfere [Circulation in the Stratosphere and Mesosphere], Leningrad, Gidrometeoizdat 1968.

25. Kashcheyev, B. L., Nechitaylenko, V. A., and Suvorov, Yu. I., "Drift of Meteor Trails," Geomagnetizm i aeronomiya 6(4), (1966).

26. Kivganov, A. F., "Radiative Sources and Sinks of Heat in the Stratosphere and Lower Mesosphere," Trudy TsAO (96), (1970).

27. Kidiyarova, V. G., "Variations in the Atmospheric Density at the Altitudes 25-80 km," Meteorologiya i gidrologiya (1), (1966).

28. Kolmogorov A. I., "Local Structure of Turbulence in Incompressible Viscous Fluid at Very High Reynolds Numbers," DAN SSSR 32(1), (1941).

29. Kolmogorov, A. I., "Dissipation of Energy in Locally Isotropic Turbulence," DAN SSSR 32(1), (1941).

30. Komarov, V. S., "Spatial Correlations of Temperature in the Free Atmosphere Over Certain Regions in the Northern Hemisphere," Trudy NIIAK (40), (1967).

31. Kramer, G., Matematicheskiye metody statistiki /Mathematical Methods of Statics/, Moscow, Foreign Literature Pub. House, 1948.

32. Lening, D. Kh., and Bettin, R. T., Sluchaynyye protsessy v zadachakh avtomaticheskogo upravleniya /Random Processes in Automatic Control Problems/, Moscow, Foreign Literature Publishing House, 1958.

33. Li, R., Optimal'nyye otsenki, opredeleniye kharakteristik i upravleniye /Optimal Estimates, Determination of Characteristics, and Control/, Moscow, "Nauka", 1966.

34. Loginov, V. F., and Sazonov, B. I., "Temperature of Northern Hemisphere and Cosmic Factors," Vestnik LGU (18)(3), (1967).

35. Lowley, D., and Maxwell, A., Faktornyy analiz kak statisticheskiy metod /Factor Analysis as a Statistical Method/, Moscow, "Nauka", 1967.

36. Mayboroda, L. A., "Use of Method of Least Integral Squares in the Problems of Analyzing the Dissipation of Nonlinear Dynamic Systems," Trudy Pervoy povolzhskoy konferentsii po avtomaticheskomu upravleniyu /Transactions of First Volga Conference on Automatic Control/, Kazan', 1971.

37. Matematicheskiye modeli i metody optimal'nogo planirovaniya /Mathematical Models and Methods of Optimal Planning/, edited by L. V. Kantorovich, Novosibirsk, 1966.

38. Mertsalova, O. B., "Methods of Calculating Vertical Correlations of Temperature and Pressure in the Free Atmosphere and Several Conclusions From the Results," Trudy NIIAK (30), (1965)

39. Mikhlin, S. G., Lektsii po nelineynym integral'nym uravneniyam /Lectures on Nonlinear Integral Equations/, Moscow, 1959.

40. Mikhnevich, V. V., Solonenko, T. A., and Filippova, A. F., "Correlation Between the Phenomenon on the Sun, Geomagnetic Disturbances, and Pressure in the Troposphere and Stratosphere," Tezisy doklada na Vsesoyuznoy konferentsii po nauchnym itogam MGSS /Abstract of a Report Presented at the All-Union Conference on the Scientific Results of the International Union of Geodesy and Geophysics/, Moscow, 1967.

42. Nalimov, V. V., and Chernova, N. A., Statisticheskiye metody planirovaniya ekstremal'nykh eksperimentov /Statistical Methods of Planning Extremal Experiments/, Moscow, "Nauka", 1965.

43. Nelineynaya optimizatsiya sistem avtomaticheskogo upravleniya /Nonlinear Optimization of Automatic Control Systems/, edited by V. M. Ponomareva, Moscow, "Mashinostroyeniye," 1970.

44. Novyye idei v planirovanii ekpserimenta /New Concepts in Experiments Planning/, edited by V. V. Nalimova, Moscow, "Nauka", 1969.

45. Newton, J., Kaiser, J. F., and Hull, L. A., Teoriya lineynykh sledyashchikh sistem /Theory of Linear Servo Systems/, Moscow, Fizmattiz, 1961.

46. "Detection and Recognition, Experiments Planning," Doklady II /303 Vsesoyuznogo soveshchaniya po statisticheskim metodam teorii upravleniya /Papers Presented at the Second All-Union Conference on Statistical Methods of Control Theory/, Moscow, "Nauka", 1970.

47. "Reliability of Complex Engineering Systems," Sb. trudov seminara sektsii nadezhnosti Nauchnogo soveta po kompleksnoy probleme "Kibernetika" pri Prezidiume AN SSSR /Collection of Works of the Seminar of the Reliability Section of the Scientific Council on the Integrated Problem "Cybernetics", Under the Presidium of the USSR Academy of Sciences/, Moscow, "Sovetskoye Radio," 1966.

48. Obukhov, A. M., "Distribution of Energy and Spectrum of Turbulent Flow," Izv. AN SSSR, ser. geogr. i geofiz. (4-5), (1941).

49. Pavlovskaya, A. A., "Relation Between Processes in the Troposphere and the Lower Stratosphere," Trudy TsIP (137), (1964).

50. Petrov, A. A., and Ryazanova, L. N., "Three Cases of Abrupt Warming of the Arctic Stratosphere," Trudy TsAO (52), (1964).

51. Pinus, N. Z., "Structure of the Wind Velocity Field in the Upper Troposphere and Lower Stratosphere," Meteorologiya i gidrologiya (4), (1962).

52. Pinus, N. A., "Results of Investigating Meso- and Microstructure of Wind Field at the Altitudes 6-12 km," Trudy TsAO (53), (1964).

53. Pinus, N. Z., and Shcherbakova, L. V., "Spectral Characteristics of Fluctuations of Wind Velocity in the Lower Half of the Troposphere," Trudy TsAO (53), (1964).

54. Piterson, I. L., Statisticheskiy analiz i optimizatsiya sistem avtomaticheskogo upravleniya /Statistical Analysis and Optimization of Automatic Control Systems/, Moscow, "Sov. radio," 1964.

55. Pogosyan, Kh. P., Obshchaya tsirkulyatsiya atmosfery /General Atmospheric Circulation/, Leningrad, Gidrometeoizdat, 1972.

56. Pogosyan, Kh. P., Pavlovskaya, A. A., and Shabel'nikova, M. V., Vzaimosvyaz' protsessov v troposfere i stratosfere severnogo polushariya /Relationship Between Processes in the Troposphere and the Stratosphere of the Northern Hemisphere/, Leningrad, Gidrometeoizdat, 1965.

57. Pogosyan, Kh. P., and Pavlovskaya, A. A., "Effect of Solar Activity on Changes in Temperature and Circulation in the Stratosphere," Meteorologiya i gidrologiya (1), (1966).

58. Ponomarev, V. M., and Mayboroda, L. A., "Method of Stochastic Approximation," Izv. AN SSSR, Tekhnicheskaya kibernetika (3), (1971).

59. Ponomarev, V. M., Teoriya upravleniya dvizheniyem kosmicheskikh apparatov /Theory of the Control of Spacecraft Motion/, Moscow, "Nauka", 1965.

60. Pugachev, V. S., Teoriya sluchaynyky funtskiy /Theory of Random Functions/, Moscow, Gostekhizdat, 1962.

61. Rastrigin, L. A., Statisticheskiye metody poiska /Statistical Methods of Search/, Moscow, "Nauka", 1968.

62. Rozenvasser, Ye. N., and Yusupov, R. M., Chuvstvitel'nost' sistem avtomaticheskogo upravleniya /Sensitivity of Automatic Control Systems/, Leningrad, Energiya Press, 1969.

63. Ryazanova, L. A., "Temperature Regime in the 25-50 km Layer," Trudy TsAO (52), (1964).

64. Sadikova, Ye. V., "Photometry of Meteors," Inform. byull. MGG (21), (1960).

65. Sadikova, Ye. V., and Sherbaum, L. M., "Results of Photographic Observations of Meteors in Kiev in 1957-1961," Problemy kosmicheskoy fiziki, Meteory (1), (1966).

66. Sazanov, B. I., Vysotnyye baricheskiye obrazovaniya i solnechnaya aktivnost' /Altitude Baric Formations and Solar Activity/, Leningrad, Gidrometeoizdat, 1964.

67. Statischeskiye metody v proyektirovanii nelineynykh sistem avtomaticheskogo upravleniya /Statistical Methods in Designing Nonlinear Automatic Control Systems/, edited by B. G. Dostupov, Moscow, "Mashinostroyeniye", 1970.

68. Stel'makh, F. N., "Correlations of Temperature in the Tropopause Layer," Trudy NIIAK (40), (1967).

69. Stel'makh, F. N., "Vertical Correlations of Temperature Over the Northern Hemisphere," Trudy NIIAK (30), (1965).

70. Tatarskiy, A. I., "Methods of Studying Atmospheric Turbulence," Izv. VUZ, Radiofizika 3(4), (1960).

71. Wild, D. J., Metody poiska ekstremuma /Methods of Search for Extremum/, Moscow, "Nauka", 1967.

72. Wilkes, S., Matematicheskaya statistika /Mathematical Statistics/, Moscow, "Nauka", 1967.

73. Faddeyev, A. K., and Faddeyeva, V. N., Vychislitel'nyye metody v lineynoy algebre /Computational Methods in Linear Algebra/, Moscow, Fizmatgiz, 1963.

74. Fel'bauma, A. A., Osnovy teorii optimal'nykh avtomaticheskikh sistem /Essentials of the Theory of Optimal Automatic Systems/, Moscow, "Nauka", 1966.

75. Finin, D., Vvedeniye v teoriyu planirovaniya eksperimentov /Introduction to the Theory of Experiment Planning/, Moscow, "Nauka", 1970.

76. Hazen, E. M., Metody optimal'nykh statisticheskikh resheniy i zadachi optimal'nogo upravleniya /Methods of Optimal Statistical Solutions and Problems of Automatic Control/, Moscow, "Sovetskoye Radio," 1968. /304

77. Hicks, C., Osnovnyye printsipy planirovaniya eksperimenta /Fundamental Principles of Experiment Planning/, Moscow, "Mir", 1967.

78. Khvostikov, I. A., "Investigation of the Atmosphere With Meteorological Rockets in the USSR in the International Geophysical Year and in the International Geophysical Collaboration," Trudy TsAO (52), (1964).

79. Chernetskiy, V. I., Analiz tochnosti nelineynykh sistem upravleniya /Analysis of the Precision of Nonlinear Control Systems/, Moscow, "Mashinostroyeniye", 1968.

80. Shabel'nikova, M. V., "Role of Vertical Motions in Temperature Change in Stratospheric Warmings," Meteorologiya i gidrologiya (11), (1966).

81. Sheffe, G., Dispersionnoy analiz /Dispersion Analysis/, Moscow, Fizmatgiz, 1963.

82. Shur, G. N., "Experimental Studies of the Energy Spectrum of Atmospheric Turbulence," Trudy TsAO (43), (1962).

83. Shur, G. N., "Spectral Density of Turbulence in the Free Atmosphere Based on Aircraft Data," Trudy TsAO (53), (1964).

84. Yudin, M. I., "Theory of Turbulence and Wind Structure as Applied to the Problem of Aircraft Vibrations," Trudy NITs GUGMS /Transactions of the Scientific Research Center of the Main Administration of the Hydrometeorological Service/ (35), Ser. 1 (1946).

85. Yurgenson, A. P., "Relationship Between Thermal Instability of the Atmosphere and the Structure of Turbulent Motions," Trudy LVIKA /Transactions of the LVIK (transliterated)/ (387), (1961).

86. Booker, H. G., and Cohen, R., "A Theory of Long-Duration Meteor Echoes Based on Atmospheric Turbulence With Experimental Confirmation," J. Geophys. Res. 61(4), (1956).

87. Booker, H. G., "Turbulence in the Ionosphere With Application to Meteor Trails, Radio-star Scintillation, Auroral Radar Echoes, and Other Phenomena," J. Geophys. Res. 61(4), (1956).

88. Beyers, N. J., and Miers, B. T., "Diurnal Temperature Change in the Atmosphere Between 30 and 60 km Over the White Sands Missile Range," J. Atmos. Scien. 22(3), (1965).

89. Cole, A. E., Kantor, A. J., and Nee, P., Stratospheric Temperature Variations 25 + 0-55 Kilometers at Latitude 15° N., J. Geophys. Res. 70(20), (1965).

90. Cole, A. E., "Suggestion of a Second Isopycnic Level of 80 to 90 Kilometers Over Churchill, Canada," J. Geophys. Res. 66(9), (1961).

91. Craig, R., and Lateef, M. A., "Vertical Motion During the 1957 Stratospheric Warming," J. Geophys. Res. 67(5), 1962.

92. Finger, F. G., and Teveles, S., "The Mid-winter 1963 Stratospheric Warming and Circulation Change," J. Appl. Met. 3(1), (1964).

93. Goody, R. M., and Lindzen, R. S., "Radiative and Photochemical Processes in Mesospheric Dynamics, Part 1, Models for Radiative and Photochemical Processes," J. Atmos. Scien. 22(4), (1965).

94. Granitzny, P., "Order of Magnitude of Large-Scale Vertical Motions in the Upper Atmosphere," Anhange zu Meteorol. Geophys., Free University of Berlin, 78(7), (1967-1968).

95. Greenhow, J. S., and Neufeld, E. L., "Measurements of Turbulence in the Upper Atmosphere," Proc. Phys. Soc. 74, Part 1 (475), (1959).

96. Greenhow, J. S., and Neufeld, E. L., "Measurements of Turbulence in the 80 to 100 km Region From the Radio Observations of Meteors," J. Geophys. Res. 61(4), (1956).

97. Greenhow, J. S., "Systematic Wind Measurements at Altitude of 80-100 km Using Radio Echoes From Meteor Trails," Phil. Mag. 45(364), (1954).

98. Greenhow, J. S., "Diurnal Variations of Density and Scale Heights in the Upper Atmosphere," J. Atmos. and Terr. Phys. 18(2/3), (1960).

99. Harris, M. F., Finger, F. G., and Teveles S., "Diurnal Variations of Winds, Pressure and Temperature in the Troposphere and Stratosphere Over the Azores," J. Atmos. Scien. 19(2), (1962).

100. Jones, L. M., Peterson, J. W., Schaefer, E. J., and Schulte H. F., "Upper Air Density and Temperature: Some Variations and Abrupt Warmings in the Mesosphere," J. Geophys. Res. 64(12), (1959).

101. Kao, S. K., and Sands, E. E., "Energy Spectrum and Eddy Kinetic Energies of the Atmosphere Between Surface and 50 Kilometers," /305 J. Geophys. Res. 72(22), (1966).

102. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," Trans. ASME, J. Basic Engineering, March, 1960.

103. Kantor, A. J., and Cole, A. E., "Zonal and Meridional Winds to 120 km," J. Geophys. Res. 69(24), (1964).

104. Kantor, A. J., and Cole, A. E., "Monthly Atmospheric Structure, Surface to 80 km," J. Appl. Met. 4(2), (1965).

105. Kapp, R. R., "The Accuracy of Winds Derived by the Radar Tracking of Chaff at High Altitudes," J. Met. 17(5), (1960).

106. Kays, M., and Graig, R. A., "On the Order of Magnitude of Large Scale Vertical Motions in the Upper Stratosphere," J. Geophys. Res. 70(18), (1965).

107. Lally, V. E., and Leviton, R. L., "Accuracy of Wind Determination From the Track of a Falling Object," Air Force Surveys in Geophysics (93), GRD AFCRC, 1958.

108. Lendhard, R. W., "Variation of Hourly Winds at 35 to 65 km During One Day at Eglin Air Force Base, Florida," J. Geophys. Res. 68(1), (1962).

109. Lindzen, R. S., "The Radiative Photochemical Response of the Mesosphere to Fluctuations in Radiations," J. Atmos. Scien. 22(5), (1965).

110. Miers, B. T., "Wind Oscillations Between 30 and 60 km Over White Sands Missle Range, New Mexico," J. Atmos. Scien. 19(2), (1962).

111. Mironovitch, V., "Stratospheric-tropospheric Evolution and Geomagnetic Activity," Beitr. Phys. Atmos. 40(3), (1967).

112. Maxwell, A., "Turbulence in the Upper Atmosphere," Phil. Mag. 45(371), (1954).

113. Murgatroyd, R. J., and Goody, R. M., "Source and Sinks of Radiative Energy From 30 to 90 km," Quart. J. Royal Met. Soc. 84(361), (1958).

114. Peterson, J. W., Hansen, W. H., Mewatters, K. D., and Banfant G., "Falling Sphere Measurements Over Kwajalein," J. Geophys. Res. 70(18), (1965).

115. Quiros, R. S., "The High-Latitude Density Regime at Rocket Altitude Inferred From Observations in Opposite Hemisphere," J. Appl. Met. 5(3), (1966).

116. Sheppard, P. A., "Dynamics of Upper Atmosphere," _J. Geophys. Res._ _64_(12), (1959).

117. Sherhag, R., "Stratospheric Temperature Change and the Associated Changes in Pressure Distribution," _J. Met._ _17_(6), (1960).

118. Sherhag, R., "Variations of the Stratospheric Temperature and Related Variations of the Pressure Distribution," _J. Met._ _17_(16), (1960).

119. Wiener, N., "_The Extrapolation, Interpolation, and Smoothing of Stationary Applications_, John Wiley and Sons, Inc. New York, 1949.

120. _Data Report of Meteorological Rocket Network_, 1961-1966.